

# Some Calculable Contributions to Holographic Entanglement Entropy

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**Ling-Yan Hung, Robert C. Myers and Michael Smolkin**

*Perimeter Institute for Theoretical Physics,  
31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada*

**ABSTRACT:** Using the AdS/CFT correspondence, we examine entanglement entropy for a boundary theory deformed by a relevant operator and establish two results. The first is that if there is a contribution which is logarithmic in the UV cut-off, then the coefficient of this term is independent of the state of the boundary theory. In fact, the same is true of all of the coefficients of contributions which diverge as some power of the UV cut-off. Secondly, we show that the relevant deformation introduces new logarithmic contributions to the entanglement entropy. The form of some of these new contributions is similar to that found recently in an investigation of entanglement entropy in a free massive scalar field theory [1].

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## 1. Introduction

Entanglement entropy was first considered in the context of the AdS/CFT correspondence by Ryu and Takayanagi [2, 3]. They provided a simple conjecture for calculating holographic entanglement entropy. In the  $d$ -dimensional boundary field theory, the entanglement entropy between a spatial region  $V$  and its complement  $\bar{V}$  is given by the following expression in the  $(d+1)$ -dimensional bulk spacetime:

$$S(V) = \frac{2\pi}{\ell_{\text{P}}^{d-1}} \text{ext}_{v \sim V} [A(v)] \quad (1.1)$$

where  $v \sim V$  indicates that  $v$  is a bulk surface that is homologous to the boundary region  $V$  [4, 5]. In particular, the boundary of  $v$  matches the ‘entangling surface’  $\partial V$  in the boundary geometry. The symbol ‘ext’ indicates that one should extremize the area over all such surfaces  $v$ .<sup>1</sup> Implicitly, eq. (1.1) assumes that the bulk physics is described by (classical) Einstein gravity and we have adopted the convention:  $\ell_{\text{P}}^{d-1} = 8\pi G_{\text{N}}$ . Hence we may observe the similarity between this expression (1.1) and that for black hole entropy. While this proposal passes a variety of consistency tests, *e.g.*, see [4, 3, 6], there is no general derivation of this holographic formula (1.1). However, a derivation has recently been provided for the special case of a spherical entangling surface in [7].

One aspect of entanglement entropy (EE), which eq. (1.1) reproduces, is that this quantity diverges and so it is only well-defined with the introduction of a short-distance cut-off  $\delta$  in the boundary field theory. Generically, the leading term obeys an ‘area law,’ being proportional to  $\mathcal{A}_{d-2}/\delta^{d-2}$  where  $\mathcal{A}_{d-2}$  denotes the area of the entangling surface in the boundary theory. While the coefficient of this divergent contribution is sensitive to the details of the UV regulator, universal data characterizing the underlying field theory can be found in subleading contributions. In particular, in a conformal field theory (CFT) with even  $d$ , one typically finds a logarithmic term  $\log(R/\delta)$  where  $R$  is some macroscopic scale characterizing the size of the region  $V$ . The coefficient of this logarithmic contribution is a certain linear combination of the central charges appearing in the trace anomaly of the CFT. The precise details of the linear combination will depend on the geometry of the background spacetime and of the entangling surface [6, 3, 8, 9].

A similar class of universal contributions were recently identified in a calculation with a free massive scalar field [1].<sup>2</sup> In particular, considering the scalar field theory (in a ‘waveguide’ geometry) with an even number of spacetime dimensions, a logarithmic contribution to the EE appears with the form

$$S_{\text{univ}} = -\gamma_d \mathcal{A}_{d-2} \mu^{d-2} \log(\mu\delta), \quad (1.2)$$

where  $\mu$  is the mass of the scalar and  $\gamma_d$  is a numerical factor depending on the spacetime dimension.<sup>3</sup> While this calculation is simplified by having a free field theory, one would still characterize the mass term as being a relevant operator. That is, the mass dominates the physics of the scalar theory at low energies but it leaves the leading UV properties unchanged, *e.g.*, the area law contribution to the EE would not be affected.

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<sup>1</sup>We are using ‘area’ to denote the  $(d-1)$ -dimensional volume of  $v$ . If eq. (1.1) is calculated in a Minkowski signature background, the extremal area is only a saddle point. However, if one first Wick rotates to Euclidean signature, the extremal surface will yield the minimal area.

<sup>2</sup>See [10] for related results in  $d = 4$ .

<sup>3</sup>To be precise,  $\gamma_d = (-)^{\frac{d-2}{2}} [6(4\pi)^{\frac{d-2}{2}} \Gamma(d/2)]^{-1}$  [1].

With this perspective, one is led to examine a natural strong coupling analog of this calculation using holography. In particular, with the standard AdS/CFT dictionary [11], we can introduce a relevant deformation of the boundary field theory by turning on a (tachyonic) scalar field in the bulk. Asymptotically the bulk geometry still approaches an AdS spacetime reflecting the fact that the boundary theory still behaves as a CFT in the far UV. However, the details of the bulk geometry are changed by the back-reaction of the scalar field and so this naturally introduces the possibility that applying eq. (1.1) in this geometry will yield new universal contributions of the form given above. In fact, our holographic calculations reveal a general class of logarithmic contributions which can appear with a relevant deformation. Schematically, they take the form of an integral over the entangling surface  $\partial V$ :

$$S_{\text{univ}} = \sum_{i,n} \gamma_i(d,n) \int_{\partial V} d^{d-2}\sigma \sqrt{H} [R, K]_i^n \mu^{d-2-2n} \log \mu \delta, \quad (1.3)$$

where  $n < (d-2)/2$ ,  $\mu$  is the mass scale appearing in the coupling of the new relevant operator,  $H_{ab}$  is the induced metric on  $\partial V$  and  $[R, K]_i^n$  denotes various combinations of the curvatures with a combined dimension  $2n$ . Both the curvature of the background geometry or the extrinsic curvature of the entangling surface may enter these expressions. The coefficients  $\gamma_i$  depend on the details of the underlying field theory and provide universal information characterizing this theory. For an even-dimensional CFT, only the contributions with  $n = (d-2)/2$  appear (and in the logarithm  $\mu$  would be replaced by a scale in the geometry) and the coefficients  $\gamma_i$  are proportional to the central charges of the CFT [3, 8]. Of course, for  $n = 0$ , there is a single contribution which matches that appearing in eq. (1.2).

One feature which is typically implicit in calculations of entanglement entropy is that the QFT is in its vacuum state. However, it is further assumed that the coefficients appearing in these logarithmic contributions are ‘universal,’ including that they are independent of the state of the underlying QFT. For example, calculating the entanglement entropy in a thermal bath would yield precisely the same result as in the vacuum state. While this feature is relatively ‘obvious’ and can be confirmed with explicit calculations, a rigorous proof is lacking. The basic idea is that the properties of the state will not modify the UV properties of the theory. One of our results in the following will be to demonstrate that the logarithmic contributions are in fact independent of the state in a holographic setting. Further our analysis makes clear that the coefficient of these contributions is a local functional of the geometry of the background in which the boundary theory resides and of the entangling surface.

An overview of the paper is as follows: We begin in section 2 by demonstrating that the coefficient of any logarithmic contribution is independent of the state of the

boundary theory. Our first discussion in this section considers the boundary theory being a pure CFT but in section 2.1, we show that this result extends to the case where the boundary theory is deformed by a relevant operator. In fact our conclusion is that in general any UV divergent terms involve local functionals of the geometry and couplings of the boundary theory. An additional feature which our analysis reveals is that new universal contributions to the entanglement entropy can arise from the presence of the relevant deformation. Hence in the subsequent sections, we present some explicit examples where such logarithmic contributions are calculated. We begin in section 3 by considering flat and spherical entangling surfaces with the boundary theory in a flat background. This exercise demonstrates that, as well as terms of the form (1.2), there are also universal contributions where the mass scale of the relevant deformation combines with the curvature of the entangling surface, as shown in eq. (1.3). In section 4, we investigate the latter contributions further by considering examples where the background in which the boundary theory resides is curved, *e.g.*,  $R \times S^{d-1}$  and  $R \times H^{d-1}$ . In section 5, we extend the approach of [12] to properly identify the precise structure of these contributions in the leading case. We conclude with a discussion of our brief results, in section 6. Finally, in appendix A, we present a holographic calculation which explicitly shows that the constant terms, which are often interpreted as universal, in odd dimensions do indeed depend on the state of the boundary theory.

## 2. Universality with a Boundary CFT

In this section, we establish that the logarithmic contribution to the entanglement entropy (EE) is independent of the state of the boundary field theory in the AdS/CFT correspondence. To begin, we consider the case where the boundary theory is purely a conformal field theory. We must then also choose the boundary dimension to be even,<sup>4</sup> since for a CFT, it is only in this case that a logarithmic contribution arises in the EE. It has been shown that the coefficient of this term is related to the central charges appearing in the trace anomaly [13, 3, 8] — see also the discussion in [9]. Implicitly, the calculations establishing this connection are made in the vacuum of the corresponding CFT and so here we are establishing that, at least in a holographic framework, the result is independent of the state of the CFT. Our present observation comes as a simple extension of the holographic calculations in [6]. There our holographic calculations were able to reproduce the precise expression for the logarithmic term in

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<sup>4</sup>However, our discussion here will consider both odd and even  $d$  because both are germane when a relevant deformation is introduced in the next subsection. We comment further on the case of odd  $d$  in appendix A.

the EE for a general entangling surface in a four-dimensional CFT, matching that which was originally determined in [8].

Let us begin by denoting the spacetime dimension of the boundary theory as  $d$  and hence the dual gravity theory has  $d + 1$  dimensions.<sup>5</sup> In the AdS/CFT correspondence, the bulk geometry asymptotically approaches anti-de Sitter space for any generic state of the boundary theory. This asymptotic geometry can then be described with the Fefferman-Graham expansion, as follows [14] — see also [15]:

$$ds^2 = \frac{L^2}{4} \frac{d\rho^2}{\rho^2} + \frac{1}{\rho} g_{ij}(x, \rho) dx^i dx^j, \quad (2.1)$$

where  $L$  is the AdS curvature scale. The asymptotic boundary is approached with  $\rho \rightarrow 0$  and  $g_{ij}(x, \rho)$  admits a series expansion in the (dimensionless) radial coordinate  $\rho$

$$g_{ij}(x, \rho) = g_{ij}^{(0)}(x^i) + \rho g_{ij}^{(1)}(x^i) + \rho^2 g_{ij}^{(2)}(x^i) \\ + \cdots + \rho^{d/2} g_{ij}^{(d/2)}(x^i) + \rho^{d/2} \log \rho f_{ij}^{(d/2)}(x^i) + O(\rho^{\frac{d+1}{2}}). \quad (2.2)$$

The leading term  $g_{ij}^{(0)}$  corresponds to the metric on which the boundary CFT resides. As shown in eq. (2.2), the first few terms fall into a Taylor series expansion but this simple form breaks down at order  $\rho^{d/2}$ . In particular, for even  $d$ , a logarithmic term appears at this order while for odd  $d$ , non-integer powers of  $\rho$  begin to make an appearance — no logarithmic term appears for odd  $d$ . With this choice of coordinate system, the expectation value of the boundary stress-energy tensor becomes [15, 16]

$$\langle T_{ij} \rangle = \frac{d}{2 \ell_{\text{p}}^{d-1} L} g_{ij}^{(d/2)} + \tilde{X}_{ij} [g^{(n)}]. \quad (2.3)$$

where  $\tilde{X}_{ij}$  denotes the contribution coming from the conformal anomaly. Hence this term vanishes for odd  $d$ , while for even  $d$ , it is determined by coefficients  $g_{ij}^{(n)}$  with  $n < d/2$ .

In solving the bulk Einstein equations, both  $g_{ij}^{(0)}$  and  $g_{ij}^{(d/2)}$  can be regarded as the independent boundary data needed to determine the bulk spacetime. As noted above,

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<sup>5</sup>Our notation is essentially the same as that established in [6]. Explicitly then, our index conventions are as follows: Directions in the full (AdS) geometry are labeled with letters from the second half of the Greek alphabet, *i.e.*,  $\mu, \nu, \rho, \dots$ . Letters from the ‘second’ half of the Latin alphabet, *i.e.*,  $i, j, k, \dots$ , correspond to directions in the background geometry of the boundary CFT. Frame indices are denoted by a hat, *i.e.*,  $\hat{i}, \hat{j}$ . Meanwhile, directions along the entangling surface in the boundary are denoted with letters from the beginning of the Latin alphabet, *i.e.*,  $a, b, c, \dots$ , and directions along the corresponding bulk surface are denoted with letters from the beginning of the Greek alphabet, *i.e.*,  $\alpha, \beta, \gamma, \dots$ .

the first fixes the boundary metric while the second determines the boundary stress tensor. That is,  ${}^{(d/2)}g_{ij}(x)$  is related to the state of the boundary CFT. Further, we note that the coefficients  ${}^{(n)}g_{ij}(x)$  with  $0 < n < d/2$  are completely fixed in terms of the boundary metric  ${}^{(0)}g_{ij}$ . More precisely, by expanding the gravitational equations of motion near the boundary, one solves for each of these coefficients in terms of the lower order terms in the expansion (2.2) — see section 2.1 for more details. For example (as long as  $d > 2$ ), one finds [12]:

$${}^{(1)}g_{ij} = -\frac{L^2}{d-2} \left( R_{ij}[\bar{g}] - \frac{{}^{(0)}g_{ij}}{2(d-1)} R[\bar{g}] \right), \quad (2.4)$$

where  $R_{ij}$  and  $R$  are the Ricci tensor and Ricci scalar constructed with the boundary metric  ${}^{(0)}g_{ij}$ . An alternative approach was presented in [12]. There the authors showed that these coefficients are almost completely fixed by conformal symmetries at the boundary.<sup>6</sup> This method examines the effect of Penrose-Brown-Henneaux (PBH) transformations, the subgroup of bulk diffeomorphisms which generate Weyl transformations of the boundary metric. Consistency of the PBH transformations on the asymptotic expansion (2.2) essentially determines all of the coefficients up to order  $n < d/2$  — see section 5 for further details. This approach also makes clear that these coefficients can be expressed as covariant tensors constructed from curvatures of the boundary metric (as well as covariant derivatives of these), as illustrated in eq. (2.4). Further, one finds that the resulting expression for  ${}^{(n)}g_{ij}(x)$  contains  $2n$  derivatives.

As these coefficients in the asymptotic geometry (2.2) depend only on the boundary metric  ${}^{(0)}g_{ij}$  and are completely independent of the state of the boundary CFT, we refer to them as the ‘fixed boundary data.’ Hence, if the logarithmic contribution to the EE is independent of the state, we must demonstrate that in our holographic calculations of the EE (for even  $d$ ), this contribution depends only on this fixed boundary data and is independent of any coefficients  ${}^{(n)}g_{ij}$  with  $n \geq d/2$ .

The holographic EE is determined by evaluating eq. (1.1) and hence the structure of the result, *i.e.*, the coefficient of the logarithmic contribution, depends on the geometry of the extremal surface  $v$ . Hence, as well as the bulk geometry (2.2), we must also consider the embedding of the corresponding  $(d-1)$ -dimensional surface in the  $(d+1)$ -dimensional bulk geometry. This embedding may be described by  $X^\mu = X^\mu(y^a, \tau)$ , where  $X^\mu = \{x^i, \rho\}$  are the bulk coordinates and  $\sigma^\alpha = \{y^a, \tau\}$  are the coordinates on surface  $m$  (with  $a = 1, \dots, d-2$ ) — recall our conventions from footnote 5. The induced metric on the bulk surface is then given by

$$h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}[X], \quad (2.5)$$

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<sup>6</sup>These calculations leave some small ambiguity that must still be fixed by the equations of motion.

where  $g_{\mu\nu}$  denotes the full  $(d+1)$ -dimensional bulk metric.

The calculations below are simplified somewhat if we fix reparameterizations of the coordinates on  $v$  with the following gauge choices (as in [12])

$$\tau = \rho \quad \text{and} \quad h_{a\tau} = 0. \quad (2.6)$$

Now following [17], one finds that the remaining embedding functions  $X^i(y^a, \tau)$  are then described by the following series expansion<sup>7</sup> for small  $\rho$ :

$$\begin{aligned} X^i(y^a, \rho) = & \overset{(0)}{X}^i(y^a) + \tau \overset{(1)}{X}^i(y^a) + \tau^2 \overset{(2)}{X}^i(y^a) \\ & + \cdots + \tau^{d/2} \overset{(d/2)}{X}^i(y^a) + \tau^{d/2} \log \tau \overset{(d/2)}{Y}^i(y^a) + O(\tau^{\frac{d+1}{2}}). \end{aligned} \quad (2.7)$$

Essentially the form of this expansion matches that for the bulk metric in eq. (2.2). The first term  $\overset{(0)}{X}^i(y^a)$  describes the position of  $\partial v$  in the boundary of the asymptotically AdS space. That is, this matches the position of the entangling surface  $\partial V$  in the boundary metric  $\overset{(0)}{g}_{ij}(x)$ . Here as in eq. (2.2), the simple Taylor series expansion appearing for the first few terms breaks down at order  $\rho^{d/2}$ . Again, for even  $d$ , a logarithmic term appears at this order while for odd  $d$ , non-integer powers of  $\rho$  begin to appear — no logarithmic term appears for odd  $d$ . This expansion (2.7) is constructed in detail by solving the local equations of motion for  $X^\mu(y^a, \tau)$  which extremize the area of the bulk surface  $v$  [17] — see section 2.1 for more details. In solving these equations, the full surface is determined by independently fixing both  $\overset{(0)}{X}^i(y^a)$  and  $\overset{(d/2)}{X}^i(y^a)$ . In particular, the latter data would be chosen to ensure that the surface  $v$  closes off smoothly in the interior of the asymptotically AdS space. As the equations are solved iteratively order by order in  $\tau$ , the coefficients  $\overset{(n)}{X}^i(y^a)$  with  $n < d/2$  are completely determined as local functionals of  $\overset{(0)}{X}^i(y^a)$  and  $\overset{(0)}{g}_{ij}$ . In particular then, these terms are independent of the state of the boundary CFT. Hence in discussing the extremal surface  $m$ , we extend the meaning of the ‘fixed boundary data’ to include both  $\overset{(n)}{g}_{ij}$  and  $\overset{(n)}{X}^i(y^a)$  with  $n < d/2$ .

As a further note, we add that these leading contributions for the embedding functions can again be determined with the application of PBH transformations [12] — see section 5 for more details. We also comment that the analysis in refs. [17, 12] was more general in considering extremal bulk submanifolds ending on a boundary surface with an arbitrary dimension  $k$ . In general, they found that the second set of independent coefficients entered the expansion of the embedding functions at order

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<sup>7</sup>The gauge condition (2.6) is not the same as in [17], however, the general structure of the asymptotic expansion (2.7) remains unaltered in both cases.



$\tau^{(k+2)/2}$ . Hence it is only for  $k = d - 2$ , the case of present interest, that the form of the expansion (2.7) matches the metric expansion (2.2).

Given the expansions of the bulk metric (2.2) and the embedding functions (2.7), the induced metric (2.5), compatible with the gauge choice (2.6), can also be expanded in the vicinity of the AdS boundary

$$h_{\tau\tau} = \frac{L^2}{4\tau^2} \left( 1 + \overset{(1)}{h}_{\tau\tau} \tau + \cdots \right), \quad h_{ab} = \frac{1}{\tau} \left( \overset{(0)}{h}_{ab} + \overset{(1)}{h}_{ab} \tau + \cdots \right). \quad (2.8)$$

Note that  $\overset{(0)}{h}_{ab} = H_{ab}$ , *i.e.*, it is precisely the induced metric on the entangling surface  $\partial V$  as evaluated in the boundary CFT. A crucial feature emerging from this perturbative construction is that the coefficients  $\overset{(n)}{h}_{\alpha\beta}$  depend only on the fixed boundary data for  $n < d/2$ , *i.e.*, they are completely determined by  $\overset{(0)}{X}^i(y^a)$  and  $\overset{(0)}{g}_{ij}$ . Now in calculating the holographic EE (1.1), we must evaluate the area

$$A(v) = \int_v d^{d-1} \sigma \sqrt{h} = \int_v d^{d-2} y d\tau \frac{L}{2\tau^{d/2}} \sqrt{\det \overset{(0)}{h}} \left[ 1 + \left( \overset{(1)}{h}_{\tau\tau} + \overset{(0)}{h}^{ab} \overset{(1)}{h}_{ab} \right) \frac{\tau}{2} + \cdots \right]. \quad (2.9)$$

The integral over the radial direction  $\tau$  extends down to an asymptotic regulator surface at  $\tau_{min} = \rho_{min} = \delta^2/L^2$  where  $\delta$  is a short distance cut-off in the boundary theory. We are interested in the appearance of a logarithmic contribution of the form  $\log \delta$ , hence we must carry out the expansion in the bracketed expression to order  $\tau^{\frac{d-2}{2}}$ , which produces the term in the integral with an overall power  $1/\tau$ . The explicit term appearing at this order for general  $d$  would be quite complicated. However, for our purposes, it suffices to know that this term will involve coefficients  $\overset{(n)}{h}_{\alpha\beta}$  with  $n \leq (d-2)/2$ . Hence this logarithmic contribution to the holographic EE is completely determined by the fixed boundary data. That is, we do not require the state dependent coefficient  $\overset{(d/2)}{g}_{ij}$ , nor details of the shape of the extremal surface  $v$  beyond what is determined by the boundary geometry  $\overset{(0)}{g}_{ij}$  and  $\overset{(0)}{X}^i(y^a)$ . We also observe that, as expected, there is no term at the appropriate order in eq. (2.9) to produce a logarithmic contribution in the case of odd  $d$ . Further, our analysis above shows that all of the divergent contributions to the holographic EE will only depend on this fixed boundary data, *i.e.*, these contributions are completely determined as local functionals of  $\overset{(0)}{g}_{ij}$  and  $\overset{(0)}{X}^i(y^a)$ .

## 2.1 Universality with a Relevant Deformation

Motivated by the recent results in [1], we wish to consider how the holographic EE is modified by the introduction of a mass term in the boundary theory. For example,

a scalar mass term would deform the boundary theory by an operator of dimension  $\Delta = d - 2$  while a fermion mass term would introduce a deformation of dimension  $\Delta = d - 1$ . Our analysis here will be more general and consider modifications of holographic EE when the boundary theory is deformed by a general relevant (scalar) operator with  $\Delta < d$ . The dual of such a scalar operator will be a scalar field in the bulk. Hence our starting point is the following bulk action where we have Einstein gravity coupled to a scalar field

$$I = \frac{1}{2\ell_{\text{p}}^{d-1}} \int d^{d+1}x \sqrt{-G} \left[ R - \frac{1}{2}(\partial\Phi)^2 - V(\Phi) \right], \quad (2.10)$$

where

$$V(\Phi) = -\frac{d(d-1)}{L^2} + \frac{1}{2}m^2\Phi^2 + \frac{\kappa}{6L^2}\Phi^3 + O(\Phi^4). \quad (2.11)$$

Now a relevant operator primarily affects the IR properties of the theory but has an insignificant effect in the UV regime. In the present holographic framework then, the dual bulk geometry still approaches AdS space in the presence of the relevant operator. Hence we consider the background geometry which asymptotically approaches  $\text{AdS}_{d+1}$  in Graham-Fefferman coordinates [14], as in eq. (2.1),

$$ds^2 = \frac{L^2}{4} \frac{d\rho^2}{\rho^2} + \frac{1}{\rho} g_{ij}(x, \rho) dx^i dx^j. \quad (2.12)$$

Of course, with  $\Phi = 0$ , the vacuum solution in the bulk will be precisely  $\text{AdS}_{d+1}$ , as described in the previous section. In general, the boundary theory's metric is still given by  $g_{ij}(x, \rho = 0) = g_{ij}^{(0)}$ , however, as we will see below, the details of the small- $\rho$  expansion will change with  $\Phi \neq 0$ . If we turn on the scalar as a probe field in this background, the scalar has two independent solutions asymptotically [11]

$$\Phi \simeq \rho^{\Delta_-/2} \phi^{(0)} + \rho^{\Delta_+/2} \phi^{(\Delta - \frac{d}{2})}, \quad (2.13)$$

where

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 L^2}. \quad (2.14)$$

The standard approach is to interpret the conformal dimension of the dual operator as  $\Delta = \Delta_+$ . Then, the leading coefficient of the more slowly decaying solution,  $\phi^{(0)}$ , is interpreted as the coupling for the dual operator in the boundary theory, while  $\phi^{(\Delta - \frac{d}{2})}$  yields the expectation value of this operator as [18]<sup>8</sup>

$$\langle \mathcal{O}(x) \rangle = (2\Delta - d) \phi^{(\Delta - \frac{d}{2})}(x). \quad (2.15)$$

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<sup>8</sup>In general, additional contributions may appear on the right-hand side involving  $\phi^{(n)}$  with  $n < \Delta - \frac{d}{2}$ . These terms are related to contact terms in correlation functions of  $\mathcal{O}$  with itself and with the stress-energy tensor [15].

Since we wish to consider a relevant operator in the boundary theory, eq. (2.14) requires that  $m^2 < 0$  for the bulk scalar and then, in fact, both of the modes in eq. (2.13) decay asymptotically (as  $\rho \rightarrow 0$ ). Note that this approach allows us to study  $\Delta \geq d/2$  and we must an ‘alternative quantization’ to study operators of lower dimension [18]. We return to this issue in the discussion in section 6.

While eq. (2.13) was derived in the probe approximation, we will see below that the powers appearing in the asymptotic scalar expansion do not change when we consider the fully back-reacted solution. The latter arises because we are considering a relevant operator with  $\Delta < d$ .<sup>9</sup> Hence even in the full nonlinear analysis of the equations of motion,  $\Phi \rightarrow 0$  as  $\rho \rightarrow 0$  and so the scalar remains a small perturbation asymptotically. Hence the leading power in the small- $\rho$  expansion of the scalar field is  $\rho^{\alpha/2}$  with  $\alpha = \Delta_-$  and

$$0 < \alpha \leq \frac{d}{2}. \quad (2.16)$$

Here, the upper limit on  $\alpha$  is the BF bound while the lower limit is simply the requirement that  $\Delta < d$ .

Let us now turn to holography with the fully back-reacted solutions of the Einstein-scalar theory (2.10). This construction is discussed in some detail in [15] and we follow their discussion below. The Einstein equations can be expressed as

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{d-1}G_{\mu\nu}V(\Phi), \quad (2.17)$$

and the scalar wave equation is

$$\frac{1}{\sqrt{-G}}\partial_\mu\left(\sqrt{-G}G^{\mu\nu}\partial_\nu\Phi\right) - \frac{\delta V}{\delta\Phi} = 0. \quad (2.18)$$

Now we write the scalar field as

$$\Phi(x, \rho) = \rho^{\alpha/2}\phi(x, \rho) \quad (2.19)$$

where we have extracted the leading asymptotic decay for this field. Combining this

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<sup>9</sup>In the case of a marginal operator with  $\Delta = d$ , the boundary theory remains a CFT and so the results regarding the holographic EE are unchanged from our previous discussion.

form with the metric ansatz (2.12), the Einstein equations (2.17) yield [15]

$$\begin{aligned}
& \rho \left( 2g''_{ij} - 2g'_{ik}g'^{kl}g'_{lj} + g'^{kl}g'_{kl}g'_{ij} \right) + L^2 R_{ij}[g] - (d-2)g'_{ij} - g'^{kl}g'_{kl}g_{ij} = \\
& = \frac{\rho^\alpha}{2} \left( L^2 \partial_i \phi \partial_j \phi + \frac{g_{ij}}{(d-1)\rho} \left( m^2 L^2 \phi^2 + \frac{\kappa}{3} \rho^{\alpha/2} \phi^3 + O(\rho^\alpha \phi^4) \right) \right) \\
& \nabla_i (g'^{kl}g'_{kl}) - \nabla^k g'_{ki} = \rho^\alpha L \left( \phi' \partial_i \phi + \frac{\alpha}{2\rho} \phi \partial_i \phi \right) \\
& g'^{kl}g''_{kl} - \frac{1}{2}g'^{ij}g'_{jk}g'^{kl}g'_{li} = \rho^\alpha \left( \phi'^2 + \frac{\alpha}{\rho} \phi \phi' + \frac{\alpha^2}{4\rho^2} \phi^2 \right. \\
& \quad \left. + \frac{1}{4(d-1)\rho^2} \left( m^2 L^2 \phi^2 + \frac{\kappa}{3} \rho^{\alpha/2} \phi^3 + O(\rho^\alpha \phi^4) \right) \right)
\end{aligned} \tag{2.20}$$

where the primes denote differentiation with respect to  $\rho$  and  $\nabla_i$  is the covariant derivative constructed from the metric  $g_{ij}(x, \rho)$ . Further  $R_{ij}[g]$  in the first line above denotes the Ricci tensor calculated for the  $d$ -dimensional metric  $g_{ij}(x, \rho)$ , treating  $\rho$  as an extra parameter — in particular then, this is *not* just the boundary Ricci tensor calculated with  $\overset{(0)}{g}_{ij}$ . Meanwhile the scalar wave equation becomes

$$0 = \rho\phi'' + \left( \alpha + 1 - \frac{d}{2} \right) \phi' + \frac{1}{2} \partial_\rho \log(-g) \left( \frac{\alpha}{2} \phi + \rho\phi' \right) + \frac{L^2}{4} \square_g \phi - \frac{\kappa}{8} \rho^{\frac{\alpha}{2}-1} \phi^2 + O(\rho^{\alpha-1} \phi^3). \tag{2.21}$$

Here we eliminated the leading term in this equation since vanishes for  $\alpha = d - \Delta$  or  $\Delta$ , just as in the probe analysis leading to eq. (2.13). Further, we have defined  $\square_g \phi \equiv \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi)$  where the full metric  $g_{ij}(x, \rho)$  is again inserted in this wave operator along the boundary directions.

Now we are in a position to construct solutions with a small- $\rho$  expansion near the asymptotic boundary. We should say that our objective here is primarily to understand the form of these solutions. We will delay considering explicit solutions to the next two sections. Hence, to get a feeling for the expansion of  $g_{ij}(x, \rho)$ , we consider the equations of motion (2.20) and (2.21) with

$$g_{ij}(x, \rho) \simeq \overset{(0)}{g}_{ij}(x) + \rho^\beta \overset{(\beta)}{g}_{ij}(x) \quad \text{and} \quad \phi(x, \rho) \simeq \phi^{(0)}(x) + \rho^{\beta'} \phi^{(\beta')}(x), \tag{2.22}$$

where we assume both exponents  $\beta, \beta'$  are positive. We begin by examining the first equation in eq. (2.20) to linear order in  $\overset{(\beta)}{g}_{ij}$ . First we find that there are two homogeneous solutions, *i.e.*, solving with all of the terms linear in  $g''$  and  $g'$ . That is, we have either  $\beta = 0$  or  $\beta = d/2$  where the latter also requires  $\overset{(0)}{g}{}^{ij} \overset{(\beta)}{g}_{ij} = 0$ . The first case is simply deformation of the boundary metric  $\overset{(0)}{g}_{ij}$  where as the second is the second independent solution containing the state data about the stress-energy, as in eq. (2.3).

This structure matches precisely that found for the usual FG expansion (2.2) and depends only on the asymptotic geometry approaching AdS geometry. At this linear level,  $R_{ij}^{(0)}(g)$  introduces an inhomogeneous source term requiring  $\beta = 1$ . Similarly  $\phi^{(0)}$  introduces various source terms on the right-hand side. The leading source comes from the mass term which requires  $\beta = \alpha$ , while the next source would be the cubic term which calls for  $\beta = 3/2 \alpha$ . Hence in the deformed background, the asymptotic expansion of  $g_{ij}(x, \rho)$  involves terms with two powers of  $\rho$ , namely, integer powers  $\rho^n$  as well as powers  $\rho^{m\alpha/2}$ . To simplify the general expansion in a workable form, we consider the case where  $\alpha/2$  is a rational number, *i.e.*,  $\alpha/2 = N/M$  where  $N$  and  $M$  are relatively prime. In this case, the general asymptotic expansion for the metric  $g_{ij}(x, \rho)$  can be written as

$$g_{ij}(x, \rho) = \sum_{n=0}^{N-1} \left( \rho^n g_{ij}^{(n)}(x) + \sum_{m=2}^{\infty} \rho^{n+m\frac{\alpha}{2}} g_{ij}^{(n+m\frac{\alpha}{2})}(x) \right) + \rho^{d/2} \sum_{n=0}^{N-1} \left( \rho^n g_{ij}^{(\frac{d}{2}+n)}(x) + \sum_{m=2}^{\infty} \rho^{n+m\frac{\alpha}{2}} g_{ij}^{(\frac{d}{2}+n+m\frac{\alpha}{2})}(x) \right). \quad (2.23)$$

If  $\rho^{d/2}$  appears in the series in the first line, the expansion contains a logarithmic term

$$g_{ij}(x, \rho) = \sum_{n=0}^{N-1} \left( \rho^n g_{ij}^{(n)}(x) + \sum_{m=2}^{\infty} \rho^{n+m\frac{\alpha}{2}} g_{ij}^{(n+m\frac{\alpha}{2})}(x) \right) + \rho^{d/2} \log \rho \sum_{n=0}^{N-1} \left( \rho^n f_{ij}^{(\frac{d}{2}+n)}(x) + \sum_{m=2}^{\infty} \rho^{n+m\frac{\alpha}{2}} f_{ij}^{(\frac{d}{2}+n+m\frac{\alpha}{2})}(x) \right). \quad (2.24)$$

Of course, the latter expansion with the logarithmic contribution always arises if  $d$  is even, as in the usual FG expansion (2.2). However, we note that this form can also arise in odd dimensions if the relevant operator has an appropriate dimension. For example, if  $d = 3$ , eq. (2.24) arises for  $\alpha = 3/m$  with  $m = 2, 3, 4, \dots$ , which would correspond to  $\Delta = 3(m-1)/m$ . In particular here,  $m = 3$  yields  $\Delta = 2$  which corresponds to the fermion mass term in  $d = 3$ .

We might also note some simplifications that can occur in the above expansions. In particular, if the boundary curvature vanishes, the integer powers are not required, *i.e.*, all of the coefficients with  $n > 0$  vanish — see the explicit solutions in section 3. Similarly for the case of a free scalar in the bulk or a scalar theory that is invariant under  $\Phi \rightarrow -\Phi$ , only (integer) powers of  $\rho^\alpha$  will appear, *i.e.*, all of the coefficients with  $m$  being a odd integer vanish.

Now using the test expansion (2.22), we can also examine the scalar wave equation (2.21) to linear order in  $\phi^{(\beta')}$ . Considering only the first two linear terms, we

find two homogeneous solutions, namely  $\beta' = 0$  and  $\beta' = d/2 - \alpha$ . The first case is simply a shift of the boundary coupling  $\phi^{(0)}$ , whereas the second power corresponds to the second independent solution. The overall power of this contribution is then  $\rho^{\frac{\alpha}{2} + \beta'} = \rho^{\frac{d-\alpha}{2}} = \rho^{\Delta_+/2}$ . Hence we see again that this structure matches precisely that found with the probe analysis in eq. (2.13), which depends only on the fact that the asymptotic geometry approaches the AdS geometry. Further, as before, the second solution contains the expectation value of the corresponding boundary theory operator, as in eq. (2.15). Continuing with the linearized analysis,  $\phi^{(0)}$  introduces two inhomogeneous source terms in eq. (2.21) from the derivative term  $\square_g \phi$  and from the higher order contributions from the scalar potential. The former requires  $\beta' = 1$  while the latter requires  $\beta' = \alpha/2$ . Hence in the full nonlinear solution, we see that the asymptotic expansion of  $\phi(x, \rho)$  also involves two powers of  $\rho$ , namely integer powers of  $\rho$  and  $\rho^{\alpha/2}$  separately, as in the metric expansion above. Given the expansions (2.23) and (2.24), we see the metric also feeds in source terms with both kinds of powers from the term with  $\partial_\rho \log(-g)$ . If we simplify the general expansion with the choice  $\alpha/2 = N/M$  as above, we find

$$\begin{aligned} \Phi(x, \rho) = \rho^{\alpha/2} \phi(x, \rho) &= \rho^{\alpha/2} \sum_{n=0}^{N-1} \sum_{m=0}^{\infty} \rho^{n+m\frac{\alpha}{2}} \phi^{(n+m\frac{\alpha}{2})}(x) \\ &+ \rho^{(d-\alpha)/2} \sum_{n=0}^{N-1} \sum_{m=0}^{\infty} \rho^{n+m\frac{\alpha}{2}} \phi^{(\frac{d}{2}-\alpha+n+m\frac{\alpha}{2})}(x). \end{aligned} \quad (2.25)$$

In this case when  $d$  is even or when  $d$  is odd and  $\alpha = (d - 2n)/(m + 2)$  (subject to the condition  $\alpha > 0$ ), the powers in the second series actually overlap with those in the first. Hence in this case, the second independent solution actually has an extra factor of  $\log \rho$ , which gives rise to the following expansion

$$\begin{aligned} \Phi(x, \rho) = \rho^{\alpha/2} \phi(x, \rho) &= \rho^{\alpha/2} \sum_{n=0}^{N-1} \sum_{m=0}^{\infty} \rho^{n+m\frac{\alpha}{2}} \phi^{(n+m\frac{\alpha}{2})}(x) \\ &+ \rho^{(d-\alpha)/2} \log \rho \sum_{n=0}^{N-1} \sum_{m=0}^{\infty} \rho^{n+m\frac{\alpha}{2}} \psi^{(\frac{d}{2}-\alpha+n+m\frac{\alpha}{2})}(x). \end{aligned} \quad (2.26)$$

Further the leading coefficient  $\psi^{(\frac{d}{2}-\alpha)}(x)$  can be related to matter conformal anomalies [15, 20].

Now before leaving our discussion of the back-reacted bulk solution, we wish to comment on the fixed boundary data. While the details of the small- $\rho$  expansion in the metric (2.12) have changed, the second independent solution still arises at order  $\rho^{d/2}$  with the coefficient  ${}^{(d/2)}g_{ij}$ . As before, this coefficient carries information about the state

of the boundary field theory through the relation in eq. (2.3). Similarly, the second independent solution appears in the expansion of the scalar field at order  $\rho^{(d-\alpha)/2}$ , which again is determined by the state of the boundary theory through eq. (2.15). Hence we may ask at what order the state data for the scalar, *i.e.*,  $\phi^{(\frac{d}{2}-\alpha)}$ , begins to contribute to the expansion of the metric. Examining the Einstein equation (2.20), we see the leading contribution will come from the mass term on the right-hand side with a cross term  $\phi^{(0)}\phi^{(\frac{d}{2}-\alpha)}$ . However, this contribution enters with a factor  $\rho^{d/2-1}$  and so we can see that  $\phi^{(\frac{d}{2}-\alpha)}$  will only effect the coefficients in the metric expansion  $g_{ij}^{(n)}$  with  $n \geq d/2$ . Hence we can still refer to the metric coefficients with  $n < d/2$  as the fixed boundary data, as well as  $\phi^{(n)}$  with  $n < d/2 - \alpha$ , since these coefficients are all independent of the state of the boundary theory and are completely fixed by the boundary metric  $g_{ij}^{(0)}$  and the coupling  $\phi^{(0)}$ .

We note that the calculation of the holographic EE is a geometric one which relies on the details of the metric expansion (2.23) or (2.24), *i.e.*, the scalar field does not directly enter into the extremal area (1.1). Hence, as in the previous section, we would like to show that the logarithmic contribution to the holographic EE only depends on the fixed boundary data, *i.e.*,  $g_{ij}^{(n)}$  with  $n < d/2$ . Hence, we must next examine the embedding functions  $X^\mu(y^a, \tau)$  to show that this universal contribution also only depends on the geometry of the entangling surface in the boundary and is independent of the details of the bulk surface, *e.g.*, ensuring that it has a regular geometry.

The embedding functions are determined by extremizing the area of the bulk surface  $v$ . It is a straightforward exercise to show that this leads to the following (local) equation of motion

$$\frac{1}{\sqrt{h}}\partial_\alpha\left(\sqrt{h}h^{\alpha\beta}\partial_\beta X^\mu\right) + h^{\alpha\beta}\Gamma^\mu_{\nu\sigma}\partial_\alpha X^\nu\partial_\beta X^\sigma = 0, \quad (2.27)$$

where the induced metric is given by eq. (2.5) and  $\Gamma^\mu_{\nu\sigma}$  denote the usual Christoffel symbols constructed with the bulk metric  $g_{\mu\nu}$ . As before, we make the gauge choices given in eq. (2.6) and in this case, setting  $\mu = \rho = \tau$  in eq. (2.27) yields a spurious constraint, which is automatically satisfied upon solving the remaining equations for  $X^i(y^a, \tau)$ . Of course, the leading terms in a small- $\tau$  expansion of the latter are just the constant terms describing the position of the entangling surface in the boundary, *i.e.*,  $X^i(y^a, \tau) \simeq X^{(0)i}(y^a)$ . To determine the order at which a second independent solution appears, we follow the analysis in [17]. We begin by assuming the equations (2.27) have been solved perturbatively to  $\mathcal{O}(\tau^s)$ , such that the coefficients  $X^{(s)i}(y^a)$  are to be solved in terms of the previous terms. One can see that the leading contribution of these new coefficients always comes from the terms in eq. (2.27) with two  $\tau$  derivatives. If we

substitute in the leading form of the induced metric (2.8), this term is given by

$$\begin{aligned}
0 &\simeq \frac{4}{L^2} \frac{\tau^{d/2}}{\sqrt{h^{(0)}}} \partial_\tau \left( \sqrt{h^{(0)}} \tau^{1-\frac{d}{2}} \partial_\tau \left( \tau^s X^i(y^a) \right) \right) + \frac{4\tau^2}{L^2} g^{ik} \partial_\tau g_{kj} \partial_\tau \left( \tau^s X^j(y^a) \right) + \dots \\
&\simeq s \left( s - \frac{d}{2} \right) \tau^{s-1} X^i(y^a) + \dots, \tag{2.28}
\end{aligned}$$

Now the latter result implies that  $X^i$  becomes undetermined for  $s = d/2$  and it can be independently specified, *e.g.*, to ensure that the extremal surface has a regular geometry. Note that this is precisely the same order at which this additional information entered in the previous section (without the relevant deformation). Further, this is also precisely the order at which the second independent set of coefficients appear in the small- $\tau$  expansion of the bulk metric. If we examine the full equations of motion (2.27) for the embedding functions in more detail, we also find that, just as in the expansions for the metric and the scalar, there are two powers of  $\tau$  ( $= \rho$ ) appear, namely, powers of  $\tau$  and  $\tau^{\alpha/2}$ . Further, it is straightforward to show that the small- $\tau$  expansion for  $X^i(y, \tau)$  takes an analogous form as that presented for the bulk metric in eq. (2.23) (or eq. (2.24), depending on the precise values of  $d$  and  $\alpha$ ). Of course, the leading coefficients  $X^i$  with  $n < d/2$  are completely determined as local functionals of  $X^i(y^a)$ ,  $g_{ij}^{(0)}$  and  $\phi^{(0)}$ .

Combining the asymptotic boundary expansions for the bulk metric and the embedding functions, one produces a similar expansion for the induced metric (2.5), *e.g.*,

$$\begin{aligned}
h_{ab}(y, \tau) &= \frac{1}{\tau} \left[ \sum_{n=0}^{N-1} \left( \tau^n h_{ab}^{(n)}(y) + \sum_{m=2}^{\infty} \tau^{n+m\frac{\alpha}{2}} h_{ab}^{(n+m\frac{\alpha}{2})}(y) \right) \right. \\
&\quad \left. + \tau^{d/2} \sum_{n=0}^{N-1} \left( \tau^n h_{ab}^{(\frac{d}{2}+n)}(y) + \sum_{m=2}^{\infty} \tau^{n+m\frac{\alpha}{2}} h_{ab}^{(\frac{d}{2}+n+m\frac{\alpha}{2})}(y) \right) \right]. \tag{2.29}
\end{aligned}$$

Of course, if  $\tau^{d/2}$  appears in the series in the first line above, then a logarithmic factor will appear in the second line, as in eq. (2.24). Recall that the leading coefficient  $h_{ab}^{(0)}$  is the induced metric  $H_{ab}$  on the entangling surface  $\partial V$  in the background for the boundary theory. The component  $h_{\tau\tau}(y, \tau)$  has an analogous expansion with a pre-factor  $L^2/(4\tau^2)$  and  $h_{\tau\tau}^{(0)} = 1$ , as in eq. (2.8). For the present purposes, an essential feature of the induced metric is that all of the coefficients  $h_{\alpha\beta}^{(n)}$  depend only on the fixed boundary data for  $n < d/2$ . That is, these leading coefficients are again completely determined by  $X^i(y^a)$ ,  $g_{ij}^{(0)}$  and  $\phi^{(0)}$ .



Now turning to the holographic EE (1.1), we must evaluate the area of the extremal surface. The area integral has the same basic structure as given in eq. (2.9) in the absence of a relevant deformation. In particular, the leading expression provides a factor of  $\tau^{-d/2}$  and the radial integral ends at the regulator surface with  $\tau_{min} = \delta^2/L^2$  where  $\delta$  is the UV cut-off in the boundary theory. We are again primarily interested in the contribution proportional to  $\log \delta$  and so we must expand the rest of the integrand to order  $\tau^{\frac{d-2}{2}}$ . While the details of this expansion are now modified by the presence of the relevant deformation, as before, it suffices to observe that a term at the desired order will only contain the coefficients  $h_{\alpha\beta}^{(n)}$  with  $n \leq (d-2)/2$ . Hence this logarithmic contribution to the holographic EE is completely determined by the fixed boundary data. That is, this contribution is completely determined by  $X^{(0)i}(y^a)$ ,  $g_{ij}^{(0)}$  and  $\phi^{(0)}$ . In fact, the same result also applies for all of the divergent contributions to the holographic EE.

While this conclusion has been unchanged by the introduction of a relevant operator in the boundary theory, the appearance (or not) of a universal contribution in the holographic EE, proportional to  $\log \delta$ , depends very much on the details, *i.e.*, on the dimension of the operator, as well as the spacetime dimension. In particular, in the expansion of the integrand in eq. (2.9), we must identify a higher order term with  $\tau^{n+m\frac{\alpha}{2}} = \tau^{\frac{d-2}{2}}$ . Of course, as in the previous section, one finds such terms with  $m = 0$  for any even  $d$ . However, there can now be new terms for odd or even  $d$  if the operator dimension of the relevant deformation is appropriate. For example, choosing an operator with  $\Delta = \frac{d}{2} + 1$  yields  $\alpha = (d-2)/2$  and hence we find the desired power of  $\tau$  with  $n = 0$  and  $m = 2$ . Similarly, for a scalar mass term with  $\Delta = d-2$ , one finds  $\alpha = d - \Delta = 2$  and hence a logarithmic term appears in even dimensions with  $d \geq 6$ .<sup>10</sup> One can compare this to the results of [1], where an analogous contribution (1.2) was found for a free massive scalar field. As a final note here, we observe that in certain instances (with appropriate  $\alpha$  and  $d$ ) the logarithmic term will be produced with both  $n$  and  $m$  nonvanishing. In the following sections, we present some explicit calculations of these new universal contributions to the holographic EE.

### 3. New Universal Terms with a Deformed Boundary Theory

While our general discussion above indicated that a relevant deformation of the boundary theory may lead to new universal contributions in the holographic EE, we would

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<sup>10</sup>Recall that  $m \geq 2$  (or  $m = 0$ ) because the stress tensor for the Einstein-scalar theory in the bulk is at least quadratic in the scalar field, *e.g.*, all of the contributions on the right-hand side of eq. (2.20) are quadratic or higher order in the scalar.

like to present some explicit examples where such logarithmic contributions appear. To make the problem of explicitly calculating the new universal terms simpler, we begin by considering here a background with a flat boundary metric. For convenience, we also depart from the conventions of the previous section by choosing a new radial coordinate  $z$  where  $\rho = z^2/L^2$ . We take the Einstein-scalar theory in eq. (2.10) with

$$V(\Phi) = -\frac{d(d-1)}{L^2} + \frac{1}{2}m^2\Phi^2 + \frac{\kappa}{6L^2}\Phi^3, \quad (3.1)$$

*i.e.*, we choose the potential to include only terms up to cubic in the scalar. Now with a flat boundary, our asymptotically AdS <sub>$d+1$</sub>  bulk metric becomes

$$ds^2 = \frac{L^2}{z^2} (dz^2 + f(z) \eta_{ij} dx^i dx^j), \quad (3.2)$$

and following eq. (2.19), we write the scalar profile as  $\Phi(z) = (z/L)^\alpha \phi(z)$ . Asymptotically, as  $z \rightarrow 0$ ,  $f(z) \rightarrow 1$  and  $\phi(z) \rightarrow \phi^{(0)}$ .

Examining the Einstein equations of motion (2.17), there are two nontrivial equations which may be written:

$$\frac{d(d-1)}{2} \left[ \left( \frac{f'}{f} \right)^2 - \frac{4}{z} \frac{f'}{f} \right] - \Phi'^2 + \frac{(mL)^2}{z^2} \Phi^2 + \frac{\kappa}{3z^2} \Phi^3 = 0, \quad (3.3)$$

$$2(d-1) \left[ f'' - \frac{d-1}{z} f' + \frac{d-4}{4} \frac{f'^2}{f} \right] + f \left( \Phi'^2 + \frac{(mL)^2}{z^2} \Phi^2 + \frac{\kappa}{3z^2} \Phi^3 \right) = 0. \quad (3.4)$$

However, the Bianchi identity ensures that these equations (combined with the scalar field equation) are redundant. For simplicity we focus on eq. (3.3) in the following. The scalar field equation (2.18) reduces to

$$\Phi'' - \frac{d-1}{z} \Phi' + \frac{d}{2} \frac{f'}{f} \Phi' - \frac{(mL)^2}{z^2} \Phi - \frac{\kappa}{2z^2} \Phi^2 = 0. \quad (3.5)$$

Now constructing power series solutions for  $f$  and  $\phi$  around  $z = 0$ , one finds

$$\begin{aligned} f(z) &= 1 + \sum_{k=2} a_k (\phi^{(0)} (z/L)^\alpha)^k, \\ \phi(z) &= \phi^{(0)} (z/L)^\alpha + \sum_{k=2} b_k (\phi^{(0)} (z/L)^\alpha)^k. \end{aligned} \quad (3.6)$$

Note that we are being somewhat cavalier in both of these expansions since neither includes the second independent solution shown in, *e.g.*, eqs. (2.23) and (2.25). However, as we showed in the previous section, none of the terms which have been neglected will

contribute to the logarithmic contributions in the holographic EE. In comparing the above expression for  $f(z)$  with eq. (2.23), we see that the terms involving  $n \neq 0$  do not appear above. This simplification occurs because the boundary curvature vanishes.

The first few coefficients in the above expansions are explicitly determined to be

$$\begin{aligned}
a_2 &= -\frac{1}{4(d-1)}, & b_2 &= -\frac{\kappa}{2\alpha(d-3\alpha)}, \\
a_3 &= \frac{2\kappa}{9\alpha(d-1)(d-3\alpha)}, \\
b_3 &= -\frac{d\alpha}{8(d-1)(d-4\alpha)} + \frac{\kappa^2}{4\alpha^2(d-3\alpha)(d-4\alpha)}, \\
a_4 &= \frac{(3d-8)\alpha + 2d}{64(d-1)^2(d-4\alpha)} - \frac{\kappa^2(5d-17\alpha)}{32\alpha^2(d-1)(d-3\alpha)^2(d-4\alpha)}, \\
b_4 &= \frac{\kappa d(17d-65\alpha)}{72(d-1)(d-3\alpha)(d-4\alpha)(d-5\alpha)} - \frac{\kappa^3(3d-10\alpha)}{24\alpha^3(d-3\alpha)^2(d-4\alpha)(d-5\alpha)}, \\
a_5 &= \frac{\kappa((79d-200)\alpha^2 - (19d-90)d\alpha - 10d^2)}{180\alpha(d-1)^2(d-3\alpha)(d-4\alpha)(d-5\alpha)} + \frac{\kappa^3(3d-13\alpha)}{30\alpha^3(d-1)(d-3\alpha)^2(d-4\alpha)(d-5\alpha)}, \\
b_5 &= \frac{3d^2\alpha^2}{64(d-1)^2(d-4\alpha)(d-6\alpha)} - \frac{\kappa^2 d(2655\alpha^2 - 1330\alpha d + 163d^2)}{576\alpha(d-1)(d-3\alpha)^2(d-4\alpha)(d-5\alpha)(d-6\alpha)} \\
&\quad + \frac{\kappa^4(6d-25\alpha)}{96\alpha^4(d-3\alpha)^2(d-4\alpha)(d-5\alpha)(d-6\alpha)}.
\end{aligned} \tag{3.7}$$

Here, we have used eqs. (3.3) and (3.5) to determine these coefficients. As an extra check, we also explicitly checked that the above coefficients also solve eq. (3.4) to order  $(\phi^{(0)}(z/L)^\alpha)^5$ . Note that if we set  $\kappa = 0$ , the only nonvanishing coefficients are  $a_k$  with even  $k$  and  $b_k$  with odd  $k$ .

Recall that the calculation of the holographic EE (1.1) is purely a geometric one and the scalar field only effects the result through its back-reaction on the bulk metric. Hence to evaluate eq. (1.1), we need only focus on the expansion of the metric, *i.e.*, the expansion of  $f(z)$  in powers of  $z^\alpha$ , as seen in eq. (3.6). As shown in section 2.1, the universal part of holographic EE involves only the terms in the expansion up to the power just proceeding  $z^d$  ( $\simeq \rho^{d/2}$ ). Hence the expansion (3.6) of  $f(z)$  must be carried out to a maximum value of  $k$ :

$$d - \alpha \leq \alpha k_{max} < d. \tag{3.8}$$

Recall from eq. (2.16), we also have  $0 \leq \alpha \leq d/2$ . Combining this inequality (3.8) with the above expansion (3.6), we see that  $k_{max} = 2, 3, 4$  or  $5$  — for which we can read the coefficients from eq. (3.7) — is sufficient for  $d/3 \leq \alpha \leq d/2$ ,  $d/4 \leq \alpha \leq d/3$ ,

$d/5 \leq \alpha \leq d/4$  or  $d/6 \leq \alpha \leq d/5$ , respectively. In general, a given  $k_{max}$  is sufficient for  $d/(1 + k_{max}) \leq \alpha \leq d/k_{max}$ . To proceed further, we choose explicit values of  $d$  and  $\Delta$ , as well as the geometry of the entangling surface. In particular for the latter, we consider 1) two flat planes bounding a slab and 2) a spherical surface  $S^{d-2}$ .

Before proceeding with explicit calculations, we address a question about interpreting the results in terms of the boundary theory. Note that with the present conventions,  $\phi^{(0)}$  is a dimensionless parameter. However, following the standard AdS/CFT dictionary, we wish to relate this parameter in terms of the coupling to an operator with conformal dimension  $\Delta$  in the boundary theory. As such, this coupling should be dimensionful defining some mass scale with  $\phi^{(0)} \sim \mu^{d-\Delta}$ . The question is then how to make this relation more precise, *e.g.*, what scale enters on the bulk side of this equation? Of course, in the AdS/CFT correspondence, the natural scale emerging from the bulk theory is simply the AdS curvature scale yielding

$$\frac{\phi^{(0)}}{L^{d-\Delta}} = \lambda \mu^{d-\Delta}. \quad (3.9)$$

Here it is convenient to introduce a dimensionless parameter  $\lambda$ , which would control the strength of the deformation in the boundary theory.<sup>11</sup> Note that given the present framework, we can not provide a more specific relation than eq. (3.9) above. For example, if we consider a mass deformation like  $m^2\phi^2$  in the boundary field theory, we could always redefine the operator by numerical factors, *e.g.*, the dual operator could equally well be  $\phi^2$  or  $\phi^2/2$  or  $\sqrt{2}\pi\phi^2$  and then accordingly the coupling would be  $m^2$  or  $2m^2$  or  $m^2/(\sqrt{2}\pi)$ . This example illustrates that distinguishing the operator from the coupling part will not be well-defined without some additional information about the boundary theory. In fact, in certain cases, the required information may be provided by supersymmetry and knowing more details of the duality between the bulk and boundary theories. One such example would be  $N = 2^*$  theories [21], where more precise results can be obtained [22].

### 3.1 Flat entangling surfaces

In this case, we introduce two flat parallel planes as the entangling surface. Hence subsystem of interest in the boundary theory is the following slab:  $V_F = \{0 \leq x^1 \leq \ell, t = 0\}$ . The holographic EE has been calculated for this geometry in the case where the boundary theory is simply a  $d$ -dimensional CFT [2, 3]:

$$S_{\text{CFT}}(V_F) = \frac{4\pi}{d-2} \frac{L^{d-1}}{\ell_P^{d-1}} \left[ \frac{R^{d-2}}{\delta^{d-2}} - \gamma_d \frac{R^{d-2}}{\ell^{d-2}} \right], \quad (3.10)$$

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<sup>11</sup>Upon converting our results below to parameters of the boundary theory, the power of  $\lambda$  keeps track of the number factors of  $\phi^{(0)}$  appearing in the holographic calculation.

where  $\gamma_d$  is a numerical factor:  $\gamma_d = \frac{1}{2} \left( 2\sqrt{\pi} \Gamma\left(\frac{d}{2(d-1)}\right) / \Gamma\left(\frac{1}{2(d-1)}\right) \right)^{d-1}$ . Here  $R$  is a regulator length along the  $x^{2,3,\dots,d-1}$  directions, which was introduced so that the entangling surfaces have a finite area, *i.e.*,  $R^{d-2}$ . Hence for a CFT in this geometry, no logarithmic contribution appears in the EE for either even or odd  $d$ . Note that the pre-factor in the above expression can be interpreted as a central charge in the boundary theory, *e.g.*, the leading singularity in the two-point of the stress tensor is controlled by the central charge, *i.e.*,  $C_T \simeq L^{d-1}/\ell_P^{d-1}$ .

Returning to the holographic EE in the presence of a relevant deformation, we describe the bulk surface  $v$  with the profile  $z = z(x^1)$  with the boundary conditions  $z(x^1 = 0) = 0 = z(x^1 = \ell)$ . The induced metric  $h_{\alpha\beta}$  on this surface embedded into the background (3.2) is given by

$$h_{\alpha\beta} dx^\alpha dx^\beta = \frac{L^2}{z^2} \left[ (f(z) + \dot{z}^2) (dx^1)^2 + f(z) \sum_{i=2}^{d-2} (dx^i)^2 \right], \quad (3.11)$$

where ‘dot’ denotes a derivative with respect to  $x^1$ . Evaluating the area of  $v$  in the bulk then yields

$$A(v) = \int \prod_{i=1}^{d-1} dx^i \sqrt{\det h_{\alpha\beta}} = R^{d-2} L^{d-1} \int_0^\ell dx^1 \frac{f^{d/2-1}}{z^{d-1}} \sqrt{f + \dot{z}^2}. \quad (3.12)$$

We treat this expression as an action for  $z(x^1)$  to determine the profile which extremizes this area. As the action contains no explicit  $x^1$  dependence, the conjugate ‘energy’ is conserved. This conserved energy functional then yields the following equation:

$$\dot{z}^2 = \left( \frac{z_*}{z} \right)^{2(d-1)} \frac{f^d}{f_*^{d-1}} - f, \quad (3.13)$$

where  $z_*$  corresponds to the (maximum) value of  $z$  where  $\dot{z} = 0$  and  $f_* = f(z_*)$ . From the inversion symmetry,  $x^1 \rightarrow \ell - x^1$ , we must have  $\dot{z} = 0$  at  $x_1 = \ell/2$ . To determine  $z_*$ , we can integrate the above equation

$$\frac{\ell}{2} = \int_0^{z_*} dz \left( \frac{z}{z_*} \right)^{d-1} \left( \frac{f^{d-1}}{f_*^{d-1}} - \left( \frac{z}{z_*} \right)^{2(d-1)} \right)^{-1/2} f^{-1/2}. \quad (3.14)$$

With these results, we can evaluate the holographic EE as follows

$$S(V_F) = \frac{2\pi}{\ell_P^{d-1}} A(v) = 4\pi \frac{L^{d-1}}{\ell_P^{d-1}} R^{d-2} \int_\delta^{z_*} dz \frac{f^{d/2-1}}{z^{d-1}} \left[ 1 - \left( \frac{f_* z^2}{f z_*^2} \right)^{d-1} \right], \quad (3.15)$$

where we have introduced the UV regulator surface at  $z = \delta$ . We are interested in extracting a universal (logarithmic) contribution from the above expression, in the limit

$\delta \rightarrow 0$ . Therefore we expand the integrand in powers of  $z$  and evaluate only the term with  $1/z$ . In fact, the expression within the square brackets can be set to 1, since the higher order terms which it contributes in this expansion begin at  $z^{d-1}$ . Therefore the desired universal coefficient is independent of  $z_*$ . In the present notation,  $z_*$  represents the undetermined data, appearing at higher order in the embedding functions, which is specified to produce a smooth surface  $v$  in the bulk. Hence, as discussed around eq. (2.28), this data will not contribute to the universal terms in the holographic EE. Further, this observation allows us to consider the limit  $\ell \rightarrow \infty$  in which case  $z_* \rightarrow \infty$  and the expression inside the square brackets above simply reduces to 1. In this limit, we are simply calculating the entanglement entropy upon dividing the boundary theory into two (semi-infinite) regions with a single wall at  $x^1 = 0$ . In the case where the boundary theory is conformal, this limit leaves only the regulator dependent term, as shown in eq. (3.10). However, the limit leaves a more interesting result in the present case because the relevant deformation in the boundary theory has introduced a finite correlation length,  $\xi = 1/\mu$ . In particular, as we now show explicitly, the result can include a universal logarithmic contribution, of the form found in [1].

As noted above, a logarithmic contribution will only appear from a term in the small- $z$  expansion of the factor  $f^{d/2-1}$  in eq. (3.15). Further, we note that given the form (3.6) of  $f$ , this expansion only produce powers  $z^{m\alpha}$  with  $m \geq 2$ . Hence to produce a  $1/z$  term in the integrand, we must have  $\alpha = (d-2)/m$ . Note that all such values appear in the allowed range given in eq. (2.16) and the conformal dimension of the dual operator would be  $\Delta = d - \frac{d-2}{m}$ . We consider explicit examples for specific values of  $m$  below.

### 3.1.1 $m = 2$

In this case, we have  $\alpha = (d-2)/2$  and we only need the first term in the expansion (3.6) of  $f$ . Examining eq. (3.15), we find

$$S(V_F) = \frac{\pi}{2} \frac{d-2}{d-1} \frac{L^{d-1}}{\ell_p^{d-1}} \lambda^2 R^{d-2} \mu^{d-2} \log \mu \delta + \dots \quad (3.16)$$

which applies for any odd or even  $d \geq 3$ . Above, we have used eq. (3.9) to write  $(\phi^{(0)})^2/L^{d-2} = \lambda^2 \mu^{d-2}$ . We have also introduced a factor of  $\mu$  to make the argument of the logarithm dimensionless, since it is the only natural scale to appear there. We are implicitly assuming an operator arises with a specific conformal dimension which is dependent on  $d$ . However, we might note that for  $d = 4$ ,  $\Delta = 3$  which corresponds to a fermion mass term while for  $d = 6$ ,  $\Delta = 4$  which corresponds to a scalar mass term.

### 3.1.2 $m = 3$

In this case,  $\alpha = (d-2)/3$  and we must expand eq. (3.6) to order  $k_{max} = 3$ . Eq. (3.15) then yields

$$S(V_F) = -\frac{2\pi\kappa}{3(d-1)} \frac{L^{d-1}}{\ell_P^{d-1}} \lambda^3 R^{d-2} \mu^{d-2} \log \mu \delta + \dots \quad (3.17)$$

using  $(\phi^{(0)})^3/L^{d-2} = \lambda^3 \mu^{d-2}$ . This result again applies for any odd or even  $d \geq 3$ . Note that this contribution vanishes for  $\kappa = 0$ , *e.g.*, for a free bulk scalar.

### 3.1.3 $m = 4$

With  $m = 4$ ,  $\alpha = (d-2)/4$  and we expand eq. (3.6) to order  $k_{max} = 4$ . Then from eq. (3.15), we obtain

$$S(V_F) = \left[ \frac{2(3d+34)}{(d-2)(d+6)^2} \kappa^2 - \frac{(3d+8)(d-2)^2}{256(d-1)} \right] \frac{\pi}{(d-1)} \frac{L^{d-1}}{\ell_P^{d-1}} \lambda^4 R^{d-2} \mu^{d-2} \log \mu \delta + \dots \quad (3.18)$$

using  $(\phi^{(0)})^4/L^{d-2} = \lambda^4 \mu^{d-2}$ . Again, we are implicitly assuming a specific operator dimension which is dependent on  $d$  but with this assumption, the corresponding universal contribution will appear for any odd or even  $d \geq 3$ .

## 3.2 Spherical entangling surfaces

In this case, we wish to calculate the EE across a spherical surface in the boundary theory. If we define the radial coordinate as usual, *i.e.*,  $r^2 = \sum_i (x^i)^2$ , in the flat boundary geometry, then the relevant subsystem is the ball:  $V_s = \{r \leq R, t = 0\}$ . Again, the holographic EE has been calculated in this case with a conformal boundary theory and a logarithmic contribution arises for even  $d$  [2, 3]:

$$S_{\text{CFT}}(V_s) = (-)^{\frac{d}{2}-1} \frac{4\pi^{d/2}}{\Gamma(d/2)} \frac{L^{d-1}}{\ell_P^{d-1}} \log(2R/\delta) + \dots \quad (3.19)$$

In fact, this result can be calculated for any CFT without any reference to holography and it is known that the pre-factor is precisely  $(-)^{\frac{d}{2}-1} 4A$  [8, 23, 9, 7] where the  $A$  is the central charge appearing in the  $A$ -type trace anomaly [24]. This contribution (3.19) will also appear in the calculation of the holographic EE when the boundary theory is deformed by a relevant operator. However, in the following, we will focus on new contributions related to the relevant deformation.

To begin, we introduce polar coordinates  $\sum_i (dx^i)^2 = dr^2 + r^2 d\Omega_{d-2}^2$  for the boundary directions in the bulk metric (3.2). We describe the bulk surface  $v$  with a profile

$r = r(z)$  with the boundary condition  $r(z = 0) = R$ . Then induced metric on  $v$  is given by

$$h_{\alpha\beta} dx^\alpha dx^\beta = \frac{L^2}{z^2} [(f(z) r'^2 + 1) dz^2 + f(z) r^2 d\Omega_{d-2}^2] , \quad (3.20)$$

where the ‘prime’ denotes a derivative with respect to  $z$ . The desired profile is chosen to minimize the area

$$A(v) = \int dz d\Omega_{d-2} \sqrt{\det h_{\alpha\beta}} = L^{d-1} \Omega_{d-2} \int_\delta dz \frac{f^{d/2-1} r^{d-2}}{z^{d-1}} \sqrt{f r'^2 + 1} , \quad (3.21)$$

where  $\Omega_{d-2}$  denotes the area of a  $(d-2)$ -dimensional unit sphere, *i.e.*,  $\Omega_{d-2} = 2\pi^{\frac{d-1}{2}} / \Gamma(\frac{d-1}{2})$ . As before, we introduce a UV regulator surface at  $z = \delta$ .

In a pure AdS background, *i.e.*,  $f(z) = 1$ , the profile which extremizes the area (3.21) has a simple form [2, 3]

$$r(z) = \sqrt{R^2 - z^2} \equiv r_0 . \quad (3.22)$$

Unfortunately, we could not find a closed form solution in the background with a generic relevant deformation. Hence to extract the universal contribution, we can proceed by solving the corresponding Euler-Lagrange equation order by order in  $z$  and then substitute the results back into the area functional (3.21). However, to leading order in this expansion  $f(z) = 1$ , for which  $r = r_0(z)$  is an exact solution. Hence it will be convenient to organize our calculations by expanding around this profile, *i.e.*, to evaluate corrections,  $\delta r = r - r_0$ , induced by the higher order terms in  $f(z)$ .

In the discussion towards the end of section 2.1, we found that the small- $\tau$  expansion of the extremal area produced a series involving powers  $\tau^{n+m\frac{\alpha}{2}}$ . In the present notation then, we expect the small- $z$  expansion to produce terms with powers  $z^{2n+m\alpha}$ . Further, from eq. (3.21), we see that the leading term in the integrand begins with  $1/z^{d-1}$  and so to produce a logarithmic contribution the expansion must contain a term where  $2n + m\alpha = d - 2$ . In fact, for even  $d$ , one finds a term where  $m = 0$  and  $n = (d - 2)/2$  which yields the same universal contribution which appears without the relevant deformation, as shown in eq. (3.19). However, we are interested in the new contributions related to the deformation and so where  $m$  is nonvanishing — as usual, this requires  $m \geq 2$ . Hence let us consider some explicit examples for specific values of  $m$ .

### 3.2.1 $m = 2$

In this case, we only keep the first correction  $k = 2$  in the expansion of  $f$ , given in eq. (3.6). Note then that the cubic, as well as any higher order interactions in



the potential of the bulk scalar play no role. Expanding eq. (3.21) to linear order in  $\delta f = f - 1$  and  $\delta r = r - r_0$  yields

$$A(v) = L^{d-1} \Omega_{d-2} \int_{\delta} dz \frac{r_0^{d-2}}{z^{d-1}} \sqrt{r_0'^2 + 1} \left( 1 + \frac{d-2 + (d-1)r_0'^2}{2(r_0'^2 + 1)} \delta f + \dots \right) \quad (3.23)$$

Above the term linear in  $\delta r$  vanishes, as it must since it is proportional to the equations of motion for  $r_0$ . Focusing our attention on the  $\delta f$  term above, we substitute the leading term from eq. (3.6), as well as a small- $z$  expansion of  $r_0$ , which combine to yield

$$\begin{aligned} A(v) \simeq & -\frac{d-2}{8(d-1)} \Omega_{d-2} L^{d-1} R^{d-2} \int_{\delta} \frac{dz}{z^{d-1}} \left( \phi^{(0)} \frac{z^{\alpha}}{L^{\alpha}} \right)^2 \\ & \times \left( 1 - \frac{(d-4)(d-1)}{2(d-2)} \frac{z^2}{R^2} + \frac{(d-1)(d-3)(d-6)}{8(d-2)} \frac{z^4}{R^4} + \dots \right) \end{aligned} \quad (3.24)$$

Now if  $\alpha = (d-2)/2$  then only the first term in the parenthesis contributes to give a logarithmic divergence, and we obtain

$$S(V_s) = \frac{\pi}{4} \frac{d-2}{d-1} \frac{L^{d-1}}{\ell_P^{d-1}} \lambda^2 \Omega_{d-2} R^{d-2} \mu^{d-2} \log \mu \delta + \dots, \quad (3.25)$$

where we have used eq. (3.9) to write  $(\phi^{(0)})^2/L^{d-2} = \lambda^2 \mu^{d-2}$ . This result (3.25) is essentially the same as that in eq. (3.16). In our holographic construction, the similarity of the results reflects the fact that to leading order  $r_0(z) \simeq R$  is constant and there is no distinction between a flat or a spherical entangling surface. Comparing eqs. (3.16) and (3.25), it appears this universal contribution can be written in the general form:

$$S_{\text{univ}} = \frac{\pi}{4} \frac{d-2}{d-1} \frac{L^{d-1}}{\ell_P^{d-1}} \lambda^2 \mathcal{A}_{d-2} \mu^{d-2} \log \mu \delta, \quad (3.26)$$

where  $\mathcal{A}_{d-2}$  is the area of the entangling surface. Again this logarithmic term will arise for any odd or even  $d \geq 3$ , for a relevant deformation with conformal dimension  $\Delta = \frac{d}{2} + 1$ .

Given eq. (3.24), we can also begin to consider contributions arising from terms in the expansion where  $n$  is also nonvanishing. If we consider the second term in the parenthesis in eq. (3.24), we see a new logarithmic contribution will appear if  $\alpha = (d-4)/2$ . In this case,

$$S(V_s) = -\pi \frac{d-4}{8} \frac{L^{d-1}}{\ell_P^{d-1}} \lambda^2 \Omega_{d-2} (R\mu)^{d-4} \log \mu \delta + \dots \quad (3.27)$$

where we use  $(\phi^{(0)})^2/L^{d-4} = \mu^{d-4}$ , as implied by the present choice of  $\alpha$ . Of course, this result only applies for  $d \geq 5$ . As should be evident from our construction above, the

terms with nonvanishing  $n$  appear in the expansion of the area integrand because of the curvature of the sphere. That is, the background geometry has vanishing curvature, both the intrinsic and extrinsic curvatures of the entangling surface are non-vanishing here. For example, the Ricci scalar of the intrinsic geometry on the entangling surface  $S^{d-2}$  is given by  $\mathcal{R} = (d-2)(d-3)/R^2$ . This suggests that we might express the result in eq. (3.27) as an integral over the sphere (contributing a factor of  $\Omega_{d-2} R^{d-2}$ ) but the integrand would be  $\mu^{d-4}$  multiplying some appropriate combination of curvatures (contributing a factor of  $1/R^2$ ). However, given the large amount of symmetry in the present geometry, it is not possible to precisely fix that latter curvature expression. We continue to investigate this question in sections 4 and 5.

Of course, it is also possible to continue with examining higher order terms in the expansion in eq. (3.24). This would in turn lead to logarithmic contributions proportional to higher powers of curvature. Schematically, these terms would take the form

$$S_{\text{univ}} \simeq \frac{L^{d-1}}{\ell_{\text{P}}^{d-1}} \lambda^2 \Omega_{d-2} (R\mu)^{d-2-2n} \log \mu\delta, \quad (3.28)$$

for  $\alpha = (d-2-2n)/2$ . Following the discussion above, it appears that these contributions take the form of an integral over the entangling surface with a factor of  $\mu^{d-2-2n}$  multiplying some combination of curvatures contributing a factor of  $1/R^{2n}$ .

### 3.2.2 $m = 3$

In this case, we are focusing on new contributions which might come from the  $k = 3$  term in the expansion of  $f$ , given in eq. (3.6). Again we consider the linear expansion given in eq. (3.23) but substitute the  $k = 3$  term for  $\delta f$ . A new logarithmic contribution arises when we assume that  $\alpha = (d-2)/3$ . Of course, this is the same exponent that appeared in section 3.1.2 and the contribution identified here has essentially the same form as in eq. (3.17). We combine these results to write a general expression,

$$S_{\text{univ}} = -\frac{\pi\kappa}{3(d-1)} \frac{L^{d-1}}{\ell_{\text{P}}^{d-1}} \lambda^3 \mathcal{A}_{d-2} \mu^{d-2} \log \mu\delta, \quad (3.29)$$

where  $\mathcal{A}_{d-2}$  is again the area of the entangling surface. Such a logarithmic term generically appears for any odd or even  $d \geq 3$  when conformal dimension of the relevant deformation is  $\Delta = \frac{2}{3}(d+1)$ . Comparing eqs. (3.26) and (3.29), we see that these two expressions have essentially the same structure, however, the details of the overall factors differ. In particular, the present contribution depends on the cubic coupling in the potential for the bulk scalar. Hence it will vanish for a free scalar field or more generally where the bulk theory is symmetric under  $\Phi \rightarrow -\Phi$ .

As above, we can also consider higher order terms in the expansion with nonvanishing  $n$  which introduce additional factors of the curvature of the sphere (*i.e.*, factors of  $1/R^{2n}$ ) in the logarithmic contributions. Schematically these terms again take a form very similar to that found in the previous analysis. In particular, for  $\alpha = (d-2-2n)/3$ , there are universal contributions of the form

$$S_{\text{univ}} \simeq \kappa \frac{L^{d-1}}{\ell_{\text{P}}^{d-1}} \lambda^3 \Omega_{d-2} (R\mu)^{d-2-2n} \log \mu \delta, \quad (3.30)$$

similar to those in eq. (3.28). We might also note that in particular cases the universal term receives contributions from more than one of the expressions outlined above. For example, consider the special case where  $\alpha = 2$  and  $d = 8$  — note that this corresponds to  $\Delta = 6$ , which is the dimension of a scalar mass term in eight dimensions. With this choice of parameters, we satisfy both  $\alpha = (d-2)/3$  and  $\alpha = (d-4)/2$  as required for the appearance of the contributions in eqs. (3.29) and (3.27), respectively. Hence the full logarithmic term in the holographic EE combines both of these contributions with

$$S(V_s) = -\pi \frac{L^7}{\ell_{\text{P}}^7} \Omega_6 R^6 \left[ \frac{\kappa \lambda^3}{21} \mu^6 + \frac{\lambda^2 \mu^4}{2 R^2} \right] \log \mu \delta + \dots. \quad (3.31)$$

### 3.2.3 $m = 4$

In this case, we extend the expansion (3.6) of  $f$  to the order  $k = 4$  and hence for consistency, we must also expand the area (3.21) to quadratic order in  $\delta f = f - 1$ . In fact, we extend the latter expansion to quadratic order in both  $\delta f$  and  $\delta r = r - r_0$  to produce

$$\begin{aligned} A(v) = & L^{d-1} \Omega_{d-2} \int_a dz \frac{r_0^{d-2}}{z^{d-1}} \sqrt{r_0'^2 + 1} \left( 1 + \frac{d-2+(d-1)r_0'^2}{2(r_0'^2+1)} \delta f + g_0(z) \delta f^2 \right. \\ & \left. + g_1(z) \delta f \delta r + g_2(z) \delta r^2 + g_3(z) (\delta r')^2 + g_4(z) \delta f \delta r' + g_5(z) \delta r \delta r' + \dots \right), \end{aligned} \quad (3.32)$$

with

$$\begin{aligned} g_0(z) &= \frac{(d-2)(d-4) + 2(d-2)(d-3)r_0'^2 + (d-1)(d-3)r_0'^4}{8(r_0'^2+1)^2}, \\ g_1(z) &= \frac{(d-1)(d-2)r_0'^2 + (d-2)^2}{2r_0(r_0'^2+1)}, \\ g_2(z) &= \frac{(d-2)(d-3)}{2r_0^2}, \quad g_3(z) = \frac{1}{2(r_0'^2+1)^2}, \\ g_4(z) &= \frac{(d-1)r_0'^3 + dr_0'}{2(r_0'^2+1)^2}, \quad g_5(z) = \frac{(d-2)r_0'}{r_0(r_0'^2+1)}. \end{aligned} \quad (3.33)$$

Next we must solve to the extremal profile by varying the above ‘action’ with respect to  $\delta r$ . The solution of the resulting equation of motion must also satisfy the boundary condition  $\delta r(z = 0) = 0$ . To leading order in  $z$ , we find

$$\delta r = \frac{d(\alpha - 1) + 2}{8(d - 1)(1 + \alpha)(d - 2 - 2\alpha)} \left( \phi^{(0)} \frac{z^\alpha}{L^\alpha} \right)^2 \frac{z^2}{R} + \dots \quad (3.34)$$

Hence we have  $\delta r \sim z^2 \delta f$ . Therefore if we choose  $\alpha = (d - 2)/4$  only terms proportional to  $\delta f, \delta f^2$  in eq. (3.32) contribute to the logarithmic divergence. Assuming further that neither  $(d - 2)/2$  nor  $3(d - 2)/4$  is an integer, eq. (3.32) yields essentially the same result as in section 3.1.3 and we combine them into a general expression

$$S_{\text{univ}} = \left[ \frac{(3d + 34)}{(d - 2)(d + 6)^2} \kappa^2 - \frac{(3d + 8)(d - 2)^2}{512(d - 1)} \right] \frac{\pi}{(d - 1)} \frac{L^{d-1}}{\ell_{\text{p}}^{d-1}} \lambda^4 \mathcal{A}_{d-2} \mu^{d-2} \log \mu \delta, \quad (3.35)$$

where as before  $\mathcal{A}_{d-2}$  is the area of the entangling surface. Such a logarithmic term appears for any odd or even  $d \geq 3$  when  $\Delta = (3d + 2)/4$ .

If either  $(d - 2)/2$  or  $3(d - 2)/4$  is an integer, then there are extra terms arising from the expansion of the coefficient in front of  $\delta f$  in eq. (3.32). These terms are associated with the effect of intrinsic curvature of the sphere and in general will be of the form given by eqs. (3.28) and (3.30). Furthermore, by suitably changing the value of  $\alpha$ , one can also consider possible scenarios where terms involving  $\delta r$  start contributing to the universal divergence.

As a specific example, let us consider the case of a ten-dimensional CFT with  $\alpha = 2$ , which corresponds to the deformation of the CFT with a scalar mass term. Then the logarithmic divergence is given by the expression in eq. (3.35) supplemented with

$$\delta S(V_s) = \frac{\pi}{4} \frac{L^9}{\ell_{\text{p}}^9} R^8 \Omega_8 \left[ \frac{7}{2} \frac{\lambda^2}{R^4} \frac{\mu^4}{R^4} + \frac{\kappa \lambda^3}{3} \frac{\mu^6}{R^2} \right] \log \mu \delta + \dots \quad (3.36)$$

We have used eqs. (3.6), (3.7) and (3.23) to evaluate this term.

## 4. Curved boundaries

In section 3.2, we examined the holographic EE for a spherical entangling surface. Our calculations there began to illustrate an interesting interplay in the coefficient of the universal contributions between the curvature of the entangling surface and the mass scale introduced by the relevant deformation. In particular, our results suggest that various new universal contributions to the holographic EE appear where the coefficient is given by an integral over the entangling surface with a factor of  $\mu^{d-2-2n}$  multiplying

some combination of curvatures contributing a factor of  $1/R^{2n}$ . However, with the results of the previous section alone, the details of these contributions remain incomplete. That is, the precise form of the appropriate curvature factor remains unclear. Here we examine these issues further by calculating the holographic EE for various entangling surfaces when the background in which the boundary theory resides is also curved.

In particular, we consider the boundary theory on certain simple backgrounds of the form  $R^1 \times \Sigma_k$  where  $\Sigma_k$  is a maximally symmetric space, where  $k \in \{\pm 1, 0\}$  indicates the sign of the curvature. That is,  $\Sigma_+ = S^{d-1}$ ,  $\Sigma_0 = R^{d-1}$  and  $\Sigma_- = H^{d-1}$ . Further, we introduce  $R$  as the background curvature scale so that the Ricci scalar takes the form

$$R[\Sigma_k] = \frac{k(d-1)(d-2)}{R^2}. \quad (4.1)$$

Of course,  $R$  can be scaled away in the case of  $k = 0$  — the simplest choice is to set  $R = L$  in this case. The corresponding bulk metric can be written as

$$ds^2 = \frac{L^2}{z^2} (dz^2 - f_t(z) dt^2 + R^2 f_k(z) d\Sigma_k^2), \quad (4.2)$$

where

$$d\Sigma_k^2 = d\theta^2 + F_k(\theta)^2 d\Omega_{d-2}^2, \quad F_k = \begin{cases} \sin \theta, & k = 1 \\ \sinh \theta, & k = -1 \\ \theta, & k = 0 \end{cases} \quad (4.3)$$

and  $d\Omega_{d-2}^2$  is the metric on a  $(d-2)$ -dimensional unit sphere. In a pure AdS background, the two metric functions  $f_{t,k}(z)$  are given by:

$$f_t = \left(1 + k \frac{z^2}{4R^2}\right)^2 \equiv f_{t,0}, \quad f_k = \left(1 - k \frac{z^2}{4R^2}\right)^2 \equiv f_{k,0}. \quad (4.4)$$

We wish to consider the Einstein-scalar theory (2.10) with the cubic potential (3.1), as in the previous section. The Einstein equations (2.17) now yield three nontrivial components but only two of these are independent. We chose to consider the following two equations:

$$\begin{aligned} & \frac{(d-2)(d-1)}{2} \left[ \left( \frac{f'_k}{f_k} \right)^2 - \frac{4}{z} \left( \frac{f'_k}{f_k} \right) \right] + (d-2) \frac{f'_k f'_t}{f_k f_t} - 2(d-1) \frac{f'_t}{z f_t} \\ & - 2 \frac{R[\Sigma_k]}{f_k} - \Phi'^2 + \frac{(mL)^2}{z^2} \Phi^2 + \frac{\kappa}{3z^2} \Phi^3 = 0 \\ & 2(d-1) \left[ f''_k - \frac{d-1}{z} f'_k + \frac{(d-4)}{4} \frac{f_k'^2}{f_k} \right] - 2 R[\Sigma_k] \\ & + f_k \left( \Phi'^2 + \frac{(mL)^2}{z^2} \Phi^2 + \frac{\kappa}{3z^2} \Phi^3 \right) = 0 \end{aligned} \quad (4.5)$$

where  $R[\Sigma_k]$  is the Ricci scalar (4.1) of the boundary geometry. The scalar field equation (2.18) becomes

$$\Phi'' - \frac{d-1}{z}\Phi' + \frac{\Phi'}{2} \left( (d-1)\frac{f'_k}{f_k} + \frac{f'_t}{f_t} \right) - \frac{(mL)^2}{z^2}\Phi - \frac{\kappa}{2z^2}\Phi^2 = 0. \quad (4.6)$$

In order for the bulk metric (4.2) to be an asymptotically AdS solution,  $f_k$  and  $f_t$  must approach a constant (*i.e.*, 1) at the boundary  $z \rightarrow 0$ . Substituting the asymptotic form  $\Phi \sim z^\alpha$  into the scalar equation again yields the expected indicial equation,  $\alpha(\alpha - d) = (mL)^2$ , which has the two solutions  $\Delta_\pm$  given in eq. (2.14). As before, we introduce a profile for  $\Phi$  beginning with  $z^\alpha$ , where  $\alpha = \Delta_- = d/2 - \sqrt{d^2/4 + (mL)^2}$ , to describe a dual operator of dimension  $\Delta_+ = d - \alpha$ . The back-reaction on the metric, *i.e.*, in  $f_{k,t}$ , again begins at order  $z^{2\alpha}$ . However, the boundary curvature now also appears as an explicit source in the Einstein equations (4.5) and its effect begins to appear at order  $z^2$ . The asymptotic expansion of the metric thus generally take the form given in eq. (2.23).

Now we would like to compute the EE in the case where the entangling surface is an  $S^{d-2}$  at  $\theta = \theta_0$  in the metric (4.3) for the spatial geometry  $\Sigma_k$ . Hence we specify the bulk surface with a profile  $\theta(z)$  satisfying the boundary condition  $\theta(z=0) = \theta_0$ . This calculation then requires a generalization of eq. (3.21),

$$S = \frac{2\pi}{\ell_P^{d-1}} L^{d-1} R^{d-2} \Omega_{d-2} \int_\delta dz \frac{f_k^{d/2-1} F_k^{d-2}}{z^{d-1}} \sqrt{1 + f_k R^2 \theta'(z)^2}. \quad (4.7)$$

In a pure AdS background (4.4), we can find an exact solution for  $\theta(z)$ :

$$\theta(z) \equiv \theta_{k,0}(z) = \begin{cases} \cos^{-1}(\cos \theta_0 (4R^2 + z^2)/(4R^2 - z^2)) , & k = 1 \\ \cosh^{-1}(\cosh \theta_0 (4R^2 - z^2)/(4R^2 + z^2)) , & k = -1 \\ \sqrt{\theta_0^2 - z^2/R^2} , & k = 0 \end{cases} \quad (4.8)$$

We were able to find these solutions because these profiles all specify essentially the same surface in different coordinate systems of the AdS geometry. Following ref. [7], this surface corresponds to the bifurcation surface of a topological AdS black hole.

Now we follow the same procedure as in section 3.2 expanding around the pure AdS solutions. That is, we expand eq. (4.7) in powers of  $\delta f$  and  $\delta \theta$ , which are defined as

$$\delta f = f_k - f_{k,0}, \quad \delta \theta = \theta - \theta_{k,0}. \quad (4.9)$$

Note that  $f_t(z)$  does not appear in our integral (4.7) and so we need not consider perturbations of this metric function. To obtain the leading contribution in  $\phi^{(0)}$ , we

expand eq. (4.7) to leading order in  $\delta f$  and  $\delta\theta$ , which gives

$$\begin{aligned}
\delta S &\simeq 2\pi \frac{L^{d-1}}{\ell_P^{d-1}} R^{d-2} \Omega_{d-2} \int_{\delta} dz \left[ \frac{f_{k,0}^{d/2-1} F_k^{d-2}(\theta_{k,0})}{z^{d-1}} \sqrt{1 + R^2 f_{k,0} \theta_{k,0}'^2} \right. \\
&\quad \left. \times \left( \frac{d-2}{2f_{k,0}} + \frac{\theta_{k,0}'^2}{1 + R^2 f_{k,0} \theta_{k,0}'^2} \right) \right] \delta f \\
&\simeq \pi(d-2) \frac{L^{d-1}}{\ell_P^{d-1}} R^{d-2} \Omega_{d-2} F_k(\theta_0)^{d-2} \int_{\delta} \frac{dz}{z^{d-1}} \delta f \\
&\quad \times \left[ 1 - \frac{1}{2}(d-4) \left( \frac{d-1}{d-2} c_k^2 + \frac{k}{2} \right) \frac{z^2}{R^2} \right] \quad (4.10)
\end{aligned}$$

where

$$c_k = \begin{cases} \cot \theta_0, & k = 1 \\ \coth \theta_0, & k = -1 \\ \theta_0^{-1}, & k = 0. \end{cases} \quad (4.11)$$

Note that the term linear in  $\delta\theta$  vanishes in eq. (4.10) by the equations of motion (for the extremal profile in AdS space).

Following the discussion in section 2, the correction to the metric  $\delta f$  may be written as

$$\delta f = \sum_{n=0}^{N-1} \sum_{m=2}^{\infty} a_{(m,n)} (\phi^{(0)}(z/L)^{\alpha})^m (z/R)^{2n}, \quad (4.12)$$

where we have assumed the exponent  $\alpha$  has the form  $\alpha/2 = N/M$ , as in the previous analysis. Our convention to normalize the factors of  $z^{2n}$  with powers of  $R$ , rather than  $L$ , is convenient in the following but it is also a natural choice because  $R[\Sigma_k]$  appears as a source in the Einstein equations 4.5. The leading coefficient  $a_{(2,0)}$  in this expansion is unaffected by the boundary curvature and takes precisely the same value as in eq. (3.7), *i.e.*,  $a_{(2,0)} = a_2 = -1/[4(d-1)]$ .

Consider first the universal contribution from  $\delta S$  arising when  $\alpha = (d-2)/2$ , as appeared in sections 3.1.1 and 3.2.1. In this case, the leading contribution in  $\delta f$  is of order  $z^{d-2}$  and the new logarithmic term is identical to that in eq. (3.26). In particular then, this result is unaffected by the background curvature.

Next we turn to  $\alpha = (d-4)/2$ , as was considered in sections 3.1.2 and 3.2.2. In this case,  $\delta f$  begins at  $z^{d-4}$  but must be expanded up to  $z^{d-2}$  to identify the logarithmic contribution to  $\delta S$ . The equations of motion give

$$\delta f = (\phi^{(0)})^2 \left( \frac{z}{L} \right)^{2\alpha} \left[ -\frac{1}{4(d-1)} + k \frac{(d-4)(d^2 - 4d + 8)}{32(d-2)(d-1)} \frac{z^2}{R^2} \right] + \dots \quad (4.13)$$

The logarithmic contribution in eq. (4.10) then becomes

$$\begin{aligned}
S_{\text{univ}} &= \pi(d-2) \frac{L^{d-1}}{\ell_{\text{P}}^{d-1}} R^{d-4} \Omega_{d-2} F_k(\theta_0)^{d-2} \lambda^2 \mu^{d-4} \log \mu \delta \\
&\quad \times \left[ \frac{1}{2}(d-4) \left( \frac{d-1}{d-2} c_k^2 + \frac{k}{2} \right) a_{(2,0)} - a_{(2,2)} \right] \\
&= -\pi \frac{L^{d-1}}{\ell_{\text{P}}^{d-1}} \lambda^2 \mu^{d-4} \log \mu \delta \int_{S^{d-2}} d^{d-2} \sigma \sqrt{H} \\
&\quad \times \left[ \frac{(d-4)}{8(d-2)^2} (K_a^{\hat{\theta} a})^2 + \frac{(d-4)(d^2 - 2d + 4)}{32(d-1)^2(d-2)} R[\Sigma_k] \right].
\end{aligned} \tag{4.14}$$

In the second expression, we have tentatively expressed the result as an integral over the entangling surface, to illustrate the kind of general expression that we anticipate. We are denoting the induced metric on this boundary surface as  $H_{ab}$ . Note that implicitly the result contains two curvature scales,  $1/R^2$  and  $c_k^2/R^2$  and hence the integrand includes two independent curvature terms. The last term involves the Ricci scalar (4.1) of the background geometry in which the boundary theory resides. The first term involves the extrinsic curvature of the entangling surface which is given by

$$K_{ab}^{\hat{\theta}} = -t_a^i t_b^j \nabla_i n_j^{\hat{\theta}} = -\frac{c_k}{R} h_{ab}^{(0)}, \tag{4.15}$$

where  $n_j^{\hat{\theta}}$  and  $t_a^i$  are respectively the normal and tangent vectors, to the entangling surface  $\partial V$  — see [6] for further details and a full discussion of our conventions. Note that in principle, there is also an extrinsic curvature associated with the normal vector in the time direction however  $K_{ab}^{\hat{t}} = 0$  in the present case.

Hence our present calculation demonstrates that  $S_{\text{univ}}$  takes a form slightly more complicated than anticipated in the discussion in section 3.2.1. In particular, there are two independent curvature contributions, whereas we could only detect one in our calculations in the previous section. We should note however that the curvatures which we have written in eq. (4.14) are only representative. For example, we easily could replace  $(K_a^{\hat{\theta} a})^2$  by  $(d-2) K_b^{\hat{\theta} a} K_a^{\hat{\theta} b}$ . Alternatively we could use the fact that the intrinsic curvature of our entangling surface has  $\mathcal{R} \propto (1/R^2 + c_k^2/R^2)$ . Of course, when we set  $k = 0$  in eq. (4.14), the result agrees with this previous calculation in a flat background. While they are informative, unfortunately these simple examples are still too symmetric to give us enough insight to properly fix the covariant expression that describes this universal term for a general entangling surface.

## 5. PBH transformations with matter

In this section, we revisit the powerful approach developed in [12] to get a more precise



understanding of the new universal contributions to the holographic EE. Here one is able to determine essentially all of the fixed boundary data by examining their behaviour under PBH transformations, the subgroup of bulk diffeomorphisms which generate Weyl transformations in the boundary. In [12] however only pure gravity theories in the bulk are considered and so we must extend their analysis to include a bulk scalar. An essential feature of our analysis is that we must not just consider  $\phi^{(0)}$  to be a coupling constant in the boundary theory, rather we must elevate it to a field. That is, we consider  $\phi^{(0)}(x)$  to take full advantage of this approach. Just as in the pure gravity case, these calculations leave some undetermined constants that must be fixed by the equations of motion. While there are no immediate obstacles to performing a general analysis, in the following, we only work out a specific example which includes a scalar field in the bulk to illustrate the general approach.

In particular, we will focus our attention on  $\alpha = (d - 4)/2$  and completely fix the universal contribution which was identified in the previous section. We fix the metric in FG gauge as in eq. (2.12), and define

$$\tilde{g}_{ij}(\phi^{(0)}) = g_{ij} + \Delta g_{ij}(\phi^{(0)}), \quad (5.1)$$

where  $g_{ij}$  is the asymptotically AdS solution without the relevant deformation turned on, *i.e.*, before the back-reaction of the scalar field is considered. Our goal is to solve for the leading terms in the expansion of  $\phi_{(0)}$  in  $\Delta g_{ij}$ . The leading terms in the expansion of the metric and the scalar field  $\Phi$  are

$$\begin{aligned} g_{ij}(x, \rho) &= g_{ij}^{(0)} + \rho g_{ij}^{(1)} + \dots, & \Delta g_{ij} &= \rho^\alpha \left( g_{ij}^{(\alpha)} + \rho g_{ij}^{(\alpha+1)} \right) + \dots, \\ \Phi(x, \rho) &= \rho^{\frac{\alpha}{2}} \left( \phi^{(0)} + \rho \phi^{(1)} \right) + \dots. \end{aligned} \quad (5.2)$$

Now the coordinate transformations which preserve the FG gauge take the form [12],

$$\rho = \rho'(1 - 2\sigma(x')), \quad x^i = x'^i + a^i(x', \rho'), \quad (5.3)$$

to leading order in some function  $\sigma$ , where

$$a^i(x, \rho) = \frac{L^2}{2} \int_0^\rho d\rho' g^{ij}(x, \rho') \partial_j \sigma(x). \quad (5.4)$$

The form of  $a^i$  is independent of the form of the series expansion of  $g_{ij}(x, \rho)$  in  $\rho$ , which is modified from that in eq. (2.2) to the more general form in eq. (2.23) in the presence of matter back-reaction. Now following the approach of [12], we substitute the metric expansion (2.23) into eq. (5.4) and use

$$\delta G_{ij} = \frac{\delta g_{ij}(x, \rho)}{\rho} = \xi^\mu \partial_\mu G_{ij} + 2\partial_{(i} \xi^{\mu} G_{j)\mu}, \quad (5.5)$$

where

$$\xi^\rho = -2\sigma(x)\rho, \quad \xi^i = a^i(x, \rho). \quad (5.6)$$

Our notation is such that the ‘symmetrization bracket’ is defined as  $A_{(i}B_{j)} = 1/2(A_iB_j + B_iA_j)$ . This allows one to determine how each coefficient in the general expansion (2.23) transforms under a general PBH transformation.

One can tell immediately from  $\xi^\rho \partial_\rho G_{ij}$  that there is a homogeneous scaling term for each coefficient of the form

$$\delta \overset{(n)}{g} = -2\sigma(x)(n-1) \overset{(n)}{g} + \dots. \quad (5.7)$$

Since the PBH transformations reduce to Weyl rescalings in the boundary, the above implies that  $\overset{(n)}{g}_{ij}$  has conformal dimension  $2(n-1)$ . In other words, the conformal dimension can be read off from the power of  $\rho$  multiplying the coefficient of interest. Particularly,  $\overset{(0)}{g}_{ij}$  always carries conformal dimension  $-2$ , as expected.

We are interested in how  $\overset{(\alpha)}{g}_{ij}$  and  $\overset{(\alpha+1)}{g}_{ij}$  transform. Expanding  $a^i(x, \rho)$  in  $\rho$ ,

$$a^i(x, \rho) = a_{(1)}^i \rho + a_{(2)}^i \rho^2 + a_{(\alpha+1)}^i \rho^{\alpha+1} + \dots, \quad (5.8)$$

and substituting into eq. (5.5), we have

$$\begin{aligned} \delta \overset{(0)}{g}_{ij} &= 2\sigma \overset{(0)}{g}_{ij}, & \delta \overset{(1)}{g}_{ij} &= a_{(1)}^k \partial_k \overset{(0)}{g}_{ij} + 2\partial_{(i} a_{(1)}^k \overset{(0)}{g}_{j)k}, \\ \delta \overset{(\alpha)}{g}_{ij} &= -2\sigma(\alpha-1) \overset{(\alpha)}{g}_{ij}, \\ \delta \overset{(\alpha+1)}{g}_{ij} &= -2\sigma\alpha \overset{(\alpha+1)}{g}_{ij} + a_{(1)}^k \partial_k \overset{(\alpha)}{g}_{ij} + a_{(\alpha+1)}^k \partial_k \overset{(0)}{g}_{ij} + 2\partial_{(i} a_{(1)}^k \overset{(\alpha)}{g}_{j)l} + 2\partial_{(i} a_{(\alpha+1)}^k \overset{(0)}{g}_{j)k}, \end{aligned} \quad (5.9)$$

where

$$a_{(1)}^i = \frac{L^2}{2} (\overset{(0)}{g}^{-1})^{ij} \partial_j \sigma, \quad a_{(\alpha+1)}^i = -\frac{L^2}{4} (\overset{(0)}{g}^{-1} \overset{(\alpha)}{g} \overset{(0)}{g}^{-1})^{ij} \partial_j \sigma. \quad (5.10)$$

These give the Weyl transformation properties of the coefficients, with which one could in principle reconstruct the series. The building blocks in the boundary theory considered in [12] include the boundary metric, its curvature tensors and their covariant derivatives. The only extra component that we have at our disposal here is the non-trivial boundary source  $\phi^{(0)}(x)$  of conformal dimension  $\alpha = d - \Delta$  and its covariant derivatives.

The solution for  $\overset{(1)}{g}_{ij}$  is unaffected by the scalar profile and is again given by eq. (2.4). Meanwhile  $\overset{(\alpha)}{g}_{ij}$  transforms homogeneously with conformal dimension  $2\alpha - 2$ . Including  $\phi^{(0)}$  amongst our building blocks, the solution is uniquely determined as

$$\overset{(\alpha)}{g}_{ij} = c_1 (\phi^{(0)})^2 \overset{(0)}{g}_{ij}, \quad (5.11)$$

where the constant  $c_1$  is fixed by the bulk equations of motion.

Substituting  ${}^{(\alpha)}g$  back into  $a_{(\alpha+1)}^i$  and hence the transformation of  ${}^{(\alpha+1)}g$ , we find the latter must have the form

$${}^{(\alpha+1)}g_{ij} = c_1 L^2 \left( d_1 (\phi^{(0)})^2 R_{ij} + d_2 {}^{(0)}g_{ij} (\phi^{(0)})^2 R + d_3 \partial_i \phi^{(0)} \partial_j \phi^{(0)} \right. \\ \left. + d_4 \nabla_i \nabla_j (\phi^{(0)})^2 + d_5 {}^{(0)}g_{ij} \square (\phi^{(0)})^2 \right). \quad (5.12)$$

Here we have more degrees of freedom than equations, and we obtain

$$d_1 = -\frac{(d-4)(d+2d_5(d^2-8d+12))}{2(d-2)^2}, \quad d_2 = \frac{(d-8d_5(d-2))(d-4)}{4(d-2)^2(d-1)}, \\ d_3 = -\frac{2(d^2-5d+4+2d_5(d^3-11d^2+36d-36))}{(d-4)(d-2)}, \quad d_4 = \frac{1}{2} - d_5(d-6), \quad (5.13)$$

leaving  $d_5$  to be determined by equations of motion.

The transformation of the scalar field gives

$$\delta \phi^{(0)} = -\sigma \alpha \phi^{(0)}, \\ \delta \phi^{(1)} = -\sigma(\alpha+2)\phi^{(1)} + \frac{L^2}{2} ({}^{(0)}g^{-1})^{ij} \partial_i \sigma \partial_j \phi^{(0)}, \quad (5.14)$$

implying that

$$\phi^{(1)} = \frac{L^2}{2(d-2(\alpha+1))} \left( \square \phi^{(0)} - \frac{1}{2(d-1)} \phi^{(0)} R \right). \quad (5.15)$$

In general if  $n\alpha = 2$  for some integer  $n$  then an extra homogenous term  $(\phi^{(0)})^{n+1}$  could appear in  $\phi^{(1)}$  and the coefficient of this term would have to be determined from the equations of motion.

As a check, our results above were compared with those obtained from directly solving the equations of motion with  $d=6$ ,  $\alpha = (d-4)/2 = 1$  and they are completely consistent. Note also that in addition to coefficients that are exactly determined above, the coefficient  $d_5$  is over-determined by the equations of motion but may be consistently solved. Hence this serves as a non-trivial check.

With these results, we can compute the leading  $\phi^{(0)}$  contribution to the universal logarithmic term in the entanglement entropy for arbitrary boundary entangling surface. The procedure is similar to the previous section. We begin by assuming that the bulk surface in the absence of relevant perturbation is given by  $X^\mu(x^\alpha, \tau)$  — where we are again working with the gauge (2.6). The back-reaction of the scalar field then introduces changes in the background metric  $\Delta g$  and also in the minimal surface  $\delta X$ .

The former has been solved to leading order in  $\phi^{(0)}$  above. The latter however, does not contribute to the entanglement entropy to leading order because  $X^i(x^\alpha, \tau = \rho)$  extremizes the action at  $\phi^{(0)} = 0$ , a fact we have made use of already in the previous sections. Since  $\Delta g$  begins at  $\tau^\alpha = \tau^{(d-4)/2}$ , together with the measure of the minimal surface  $\sqrt{h} \sim \tau^{-d/2}$ , the leading correction to the entanglement entropy goes like  $\tau^{-2}$ . To extract the log-term, one needs to expand the remaining integrand to linear order in  $\tau$ . While we do not know the complete solution of  $X(x, \tau)$  for arbitrary asymptotically AdS background and boundary entangling surface, the linear  $\tau$  term in its asymptotic expansion is universal, completely dictated by fixed boundary data, independent of the gravity theory concerned. That is, it can also be fixed by the PBH transformations which yield [12]

$$X^i(x^\alpha, \tau) = \overset{(0)}{X}^i(x^\alpha) + \tau \overset{(1)}{X}^i(x^\alpha) + \dots, \quad \overset{(1)}{X}^i = \frac{L^2 K^i}{2(d-2)}, \quad (5.16)$$

where  $K^i$  is the trace of the extrinsic curvature of the entangling surface — see eq. (4.15).

The leading  $\phi^{(0)}$  dependence of the area of the minimal surface is then given by

$$\begin{aligned} \delta A &= \int d^{d-2} y d\tau \delta \left( \sqrt{h_{\tau\tau}(\phi^{(0)}) \det h_{ij}(\phi^{(0)})} \right) \\ &= L \int d^{d-2} y \frac{d\tau}{2\tau^{d/2}} \sqrt{\tilde{h}_{\tau\tau} \det \tilde{h}_{ab}} \left( \frac{2\tau \partial_\tau X^\mu \partial_\tau X^\nu}{\tilde{h}_{\tau\tau}} + \frac{\partial_a X^\mu \partial_b X^\nu \tilde{h}^{ab}}{2} \right) \Big|_{\phi_0=0} \Delta g_{\mu\nu}, \end{aligned} \quad (5.17)$$

where we have defined  $\tilde{h}_{\tau\tau} = 4\tau^2 h_{\tau\tau}$ ,  $\tilde{h}_{ab} = \tau h_{ab}$ , such that these quantities begin at  $O(\tau^0)$ . Further recall that the radial integral ends at the UV regulator surface  $\tau_{min} = \delta^2/L^2$ . With  $\phi^{(0)} = 0$ , the expansion of  $\tilde{h}_{\tau\tau}$  and  $\det \tilde{h}_{ij}$  are given by [12]

$$\tilde{h}_{\tau\tau} = L^2 + 4\tau (\overset{(1)}{X}^i)^2 + \dots, \quad \det \tilde{h}_{ij} = \det \overset{(0)}{h}_{ab} (1 + \tau \overset{(0)}{h}^{ab} \overset{(1)}{h}_{ab} + \dots), \quad (5.18)$$

where

$$\overset{(1)}{h}_{ab} = \overset{(1)}{g}_{ab} - \frac{L^2}{d-2} K^i K_{ab} \overset{(0)}{g}_{ij}, \quad (5.19)$$

and  $\overset{(1)}{g}_{ab}$  is as defined in eq. (2.4), but projected on to the boundary entangling surface by contracting with tangent vectors  $\partial_a \overset{(0)}{X}^i$ . We finally have

$$\begin{aligned} S_{\text{univ}} &= -2\pi \frac{L^{d-1}}{\ell_{\text{p}}^{d-1}} \lambda^2 \int d^{d-2} y \sqrt{\overset{(0)}{h}_{ab}} \left( -\frac{(d-1)(d-4)}{4(d-2)^2} c_1 K^i K^j \overset{(0)}{g}_{ij} \right. \\ &\quad \left. + \frac{(d-4)}{4L^2} c_1 \overset{(1)}{g}_a{}^a + \frac{1}{2L^2 \phi^{(0)2}} \overset{(\alpha+1)}{g}_a{}^a \right) \mu^{d-4} \log \mu \delta. \end{aligned} \quad (5.20)$$

This expression is now completely fixed when we apply

$$c_1 = -\frac{1}{4(d-1)}, \quad d_5 = \frac{1}{8}, \quad (5.21)$$

which were determined by solving the equations of motion.

We can combine the preceding results to explicitly write out this universal contribution for the case that  $\phi_{(0)}$  is a constant. With this simplification, the result (5.20) reduces to

$$S_{\text{univ}} = -\frac{(d-4)\pi}{32(d-2)^2} \frac{L^{d-1}}{\ell_p^{d-1}} \lambda^2 \mu^{d-4} \log \mu \delta \quad (5.22)$$

$$\times \int d^{d-2}y \sqrt{h_{ab}^{(0)}} \left( 4 K^i K^j g_{ij}^{(0)} + \frac{d^2+4}{d-1} R_a^a - 2d \frac{d-2}{(d-1)^2} R \right).$$

where  $R$  and  $R_a^a$  are, respectively, the background Ricci scalar and the background Ricci curvature contracted with  $H_{ij}$ , the induced metric expressed as a  $d$ -dimensional tensor:  $H_{ij} = g_{ij}^{(0)} - n_i^{\hat{i}} n_j^{\hat{j}}$ . Evaluating this expression (5.22) for the geometries considered in the previous section, we find complete agreement with eq. (4.14). However, the present result is completely general and can be applied for any background geometry and any (smooth) entangling surface.

## 6. Discussion

Our calculations have demonstrated two interesting properties about holographic entanglement entropy. First of all, the coefficient of any universal contribution which is logarithmic in the short-distance cut-off is independent of the state of the boundary theory. Secondly, when the boundary theory is deformed by turning on a relevant operator, new universal contributions appear including a class of the form found in [1].

Let us begin here with some discussion of the first result. The observation that these universal coefficients are independent of the state of the underlying field theory may seem trivial. As previously noted for an even dimensional CFT, the universal coefficients will be given by some linear combination of central charges in a general setting, even without holography. However, while our result is implicitly regarded as ‘obvious’ in discussions of EE, a rigorous proof has not been provided. In the AdS/CFT framework, we were able to make the separation of data depending on the state versus data depending on the action very explicit, even when the boundary theory is deformed by a relevant operator, and it is clear only the latter data contributes to determining the universal terms in the holographic EE.

Let us point out that there is the potential to produce a contradiction with relevant deformations with low conformal dimensions. Recall that the standard approach, described in section 2, allows us to study  $\Delta \geq d/2$ . The lower bound arises with  $m^2 = -d^2/4L^2$ , which corresponds to the Breitenlohner-Freedman bound in  $d+1$  dimensions [19]. However, the unitarity bound for a scalar operator in a  $d$ -dimensional CFT allows for  $\Delta \geq (d-2)/2$ . To study operators in the range  $d/2 \geq \Delta \geq (d-2)/2$ , we must use the ‘alternative quantization’ of the dual bulk scalar set forward in [18] for masses in the regime:

$$-\frac{d^2}{4L^2} \leq m^2 \leq -\frac{d^2}{4L^2} + \frac{1}{L^2}. \quad (6.1)$$

Hence in this regime then, we can choose the dimension of the dual operator as  $\Delta = \Delta_-$  and in this case, the roles of  $\phi^{(0)}$  and  $\phi^{(\Delta-\frac{d}{2})}$  are interchanged. We note, however, that this alternate quantization does not change the powers of  $\rho$  appearing in eq. (2.13). In particular, the leading power is still given by  $\rho^{\Delta-/2}$ . However, the key difference (for our purposes) is that the leading coefficients appearing in this asymptotic solution are now related to the state of the boundary field theory.

Hence it seems that in this situation any new universal term appearing in the holographic EE must depend on the state. However, as we now show, there is no problem because deformations in this regime do not produce any such universal contributions. Recall from our discussion in section 2.1 that a universal contribution appears in the holographic EE when, in the expansion of the integrand in eq. (2.9), a term appears with  $\tau^{n+m\frac{\alpha}{2}} = \tau^{\frac{d-2}{2}}$ . Further, recall that apart from  $m=0$ , the minimum value of  $m$  is 2 because of the structure of the Einstein-scalar theory in the bulk. This means that there is a maximum value which  $\alpha$  can have in order to produce a logarithmic contribution in the holographic EE. In particular, a logarithm will only arise for  $\alpha \leq \frac{d}{2} - 1$ . In terms of the conformal dimension of the boundary operator, this corresponds to  $\Delta \geq \frac{d}{2} + 1$  or in terms of the mass of the bulk scalar,

$$m^2 \geq -\frac{d^2}{4L^2} + \frac{1}{L^2}. \quad (6.2)$$

However, the lower limit here is interesting because comparing to eq. (6.1), we see that it precisely excludes the range of allowed masses where it is possible to make an alternate quantization of the bulk scalar.<sup>12</sup> Hence the potential problem, arising

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<sup>12</sup>The case of  $\Delta = (d-2)/2$  may still seem problematic because it corresponds to precisely the limit  $m^2 = -d^2/(4L^2) + 1/L^2$ . However, this is precisely the unitarity bound for which the dual operator is expected to be a free scalar field. However, such a CFT would be beyond the scope of the holographic models which we are considering here.

from the interchange of the roles of the different terms in the asymptotic scalar in the alternate quantization, is cleanly avoided because the deformation will simply not generate a  $\log \delta$  contribution in the holographic EE.

While the focus of our discussion has been the possible logarithmic contributions to the holographic EE, these are only the least divergent terms as  $\delta \rightarrow 0$ . The expansion of the area (2.9) will generally produce a series of terms diverging as  $1/\delta^{d-2-2n-m\alpha}$ . Of course, the first term (*i.e.*, with  $n = 0 = m$ ) yields the expected area law. Further, our analysis shows that the coefficients of all of these divergent terms are determined by the fixed boundary data in the asymptotic expansion, *i.e.*, they are all independent of the state of the boundary theory. In the present holographic framework, the general coefficient will contain  $m$  factors of the coupling  $\phi^{(0)}$  and an integral of  $n$  curvatures over the entangling surface. This observation then guarantees that the mutual information

$$I(A, B) = S(A) + S(B) - S(A \cup B) \quad (6.3)$$

for two disjoint regions  $A$  and  $B$  is free of any UV divergences in our holographic calculations. The finiteness of the mutual information is another generally accepted feature which is believed to be true in general but never rigorously proven.

We should add that implicitly we are considering a constrained class of states in this discussion,<sup>13</sup> *e.g.*, the energy density of the states being studied must be kept finite. This constraint becomes evident with the following thought experiment: Consider the boundary theory with a finite cut-off  $\delta$ , in which case it contains a finite number of degrees of freedom (if the total volume is also kept fixed). In this case, one can easily imagine choosing a state in which there is simply no entanglement between a particular region  $V$  and its complement  $\bar{V}$ . That is, we seem to have removed the potentially divergent contributions to the entanglement entropy with a particular choice of state. However, the price to be paid for this lack of correlations would be that the energy density of such a state will be of order  $1/\delta^d$ . Hence if we wish to maintain this vanishing entanglement in the limit  $\delta \rightarrow 0$ , we would require an infinite entanglement entropy. Holographically, such a state would not be dual to an asymptotic AdS geometry and so it lies outside of the class of states considered here.

In the discussion above eq. (2.8), we noted that the analysis in refs. [17, 12] provided a general analysis for bulk submanifolds with an arbitrary dimension  $k+1$ . In this case, the second set of independent coefficients appear in the expansion of the embedding functions at order  $\tau^{(k+2)/2}$ . Our analysis focussed on  $k = d - 2$ , however, for smaller values of  $k$ , the second set of free coefficients would appear at a lower order than in the expansion given in eq. (2.7). Despite appearing at a lower order, this state data does

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<sup>13</sup>RCM thanks Mark van Raamsdonk for an interesting conversation on this point.

not contribute to the coefficients of any UV divergences, in particular a logarithmic divergence, appearing in the calculation of the area of the corresponding surfaces. This occurs precisely because the dimension of the submanifold is also reduced. Hence when we evaluate the analog of eq. (2.9), the leading power becomes precisely  $\tau^{-(k+2)/2}$  and so the state dependent coefficients will only produce finite contributions to the area for general  $k$ . Hence our results extend beyond the calculation of the entanglement entropy. For example, this analysis would apply to the calculation of the expectation values of Wilson lines and shows that the coefficients of any divergent terms appearing in such a calculation are also independent of the state of the boundary theory.

In certain cases, no logarithmic contribution appears in the EE, *e.g.*, with a CFT in an odd number of spacetime dimensions. However, the constant term independent of the short-distance cut-off may then still be a universal contribution to the EE [2, 3]. The universality of this constant contribution is established for a variety of  $d = 3$  conformal quantum critical systems [25], as well as certain three-dimensional (gapped) topological phases [26]. However, from the discussion of the present paper, it is natural to expect that such a finite contribution will in fact depend on the details of the state in which the EE is calculated. Certainly in the holographic framework, this finite term should depend on  $^{(d/2)}g_{ij}$  and higher order terms in the FG expansion. We have confirmed this expectation with an explicit calculation in appendix A. Hence in general, while such a constant contribution to the EE certainly contains information with which we may characterize the underlying field theory, it will not be completely universal in the same sense as the coefficient of a logarithmic contribution. In particular then, in order to properly compare or distinguish theories with a constant contribution to EE, we must specify that this term was calculated in the vacuum state of the underlying theory.<sup>14</sup>

Of course, one of the interesting results arising from our holographic investigations was that relevant deformations of the boundary theory will produce new universal contributions to the EE, which are logarithmic in the cut-off. Schematically, the general form of the logarithmic contribution is an integral over the entangling surface  $\partial V$ :

$$S_{\text{univ}} = \sum_{i,n} \gamma_i(d,n) \int_{\partial V} d^{d-2}\sigma \sqrt{H} [R, K]_i^n \mu^{d-2-2n} \log \mu \delta, \quad (6.4)$$

where  $n < (d-2)/2$ ,  $\mu$  is the mass scale appearing in the coupling of the relevant operator,  $H_{ab}$  is the induced metric on  $\partial V$  and  $[R, K]_i^n$  denotes various combinations of the curvatures with a combined dimension  $2n$ . Both the curvature of the background

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<sup>14</sup>In general, we would have to refine further our characterization of these constant contributions to the entanglement entropy. In particular, different regulators will modify the details of the expansion the EE in terms of the cut-off and hence ambiguities should be expected to appear in the definition of the constant contribution.



geometry or the extrinsic curvature of the entangling surface may enter these expressions. The universal information which would distinguish different theories is carried in the pre-factors  $\gamma_i(d, n)$ .

As noted previously, for  $n = 0$ , we have simply  $[R, K]_1^0 = 1$  and the integral simply yields the area of the entangling surface  $\partial V$ . In this case, this contribution matches the form of the universal terms (1.2) recently found for a massive free scalar [1]. Further for  $n = (d - 2)/2$ , the above integral involves only curvatures (*i.e.*, the  $\mu$  factor reduces to one) and our expression will match the form found for an even-dimensional CFT [3, 8] — see below. More generally, the presence of these new universal terms with  $n > 0$  is easily detected with simple calculations involving symmetric geometries, as in sections 3 and 4. However, the precise form of the expressions  $[R, K]_i^n$  cannot be determined in these calculations. However, one feature that is already evident there is that the combination  $[R, K]_i^n \mu^{d-2-2n}$  appearing in the integrand has dimension  $d - 2$ , which ensures that the resulting coefficient is scale free. With the more elaborate approach outlined in section pbhmatter, the precise form of  $[R, K]_i^n$  can in principle be determined but this is a somewhat tedious exercise. Hence we have only examined the particular case of  $\alpha = (d - 4)/2$  for which the result is given in eq. (5.22). A feature of these calculations is that rather than thinking of simply a coupling constant for the relevant deformation, we must allow  $\phi^{(0)}$  to be a field which varies over the boundary geometry. This approach also highlights the connection of the entanglement entropy to a Graham-Witten anomaly [17] for the entangling surface, as noted previously for pure CFT's in [12, 6]. We should note, however, that the spacetime dimension  $d$  and the conformal dimension of the relevant operator  $\Delta$  must satisfy a particular constraint before the various terms in eq. (6.4) can appear. These constraints simply reflect the relations discussed at the end of section 2 where the logarithmic contribution appears if a term in the expansion of the area (2.9) with  $\tau^{n+m\frac{\alpha}{2}} = \tau^{\frac{d-2}{2}}$ . Hence, for a term with  $n$  to appear in eq. (6.4), we require

$$\Delta = d - \frac{d - 2 - 2n}{m} \quad \text{with integer } m \geq 2. \quad (6.5)$$

We might note that the universal contributions from relevant deformations typically appear in higher dimensions. The leading term with  $n = 0$  appears for  $d \geq 3$ . However, the terms mixing the curvatures with a power of  $\mu$  require larger values of  $d$ . For example, we require  $d \geq 5$  and  $d \geq 7$  for the contributions with  $n = 1$  and 2, respectively. Of course, as we illustrated in section 3.2.2, that a single deformation may produce more than one of these universal terms in higher dimensions, *i.e.*, with  $d = 8$  and  $\Delta = 6$ , eq. (6.5) can be satisfied with  $n = 0, m = 3$  and  $n = 1, m = 2$ .

Recall the integer  $m$  cannot be 1 above in eq. (6.5) because the stress tensor in

the bulk Einstein equations (2.17) is at least quadratic in the scalar field. We might compare this feature of our calculations to a similar result in [37]. There, a relevant operator  $\lambda \mathcal{O}$  is introduced perturbatively in a two-dimensional CFT and it is noted that this deformation only begins to have effect at order  $\lambda^2$  because the one-point function  $\langle \mathcal{O} \rangle$  vanishes in the CFT vacuum. Of course, this observation extends to CFT's in any number of dimensions and then agrees with our result that the new universal terms appear with a factor of  $\lambda^2$  or a higher power of the coupling. However, we might also contrast the differences between the two situations. First, in our holographic calculations, we are not working perturbatively, *i.e.*, we are not assuming that  $\lambda$  is small in any sense. Further, one of our key observations is that the results for the universal coefficients is independent of the state of the boundary CFT and so does not rely on calculating in the vacuum state. It would be interesting to see if in fact these features also extend beyond our holographic setting to more general CFT's.

We must emphasize that eq. (6.4) is schematic. In particular, for a given value of  $n$ , there may be several independent combinations of curvatures that appear, including both background curvatures and the extrinsic curvature of the entangling surface. Our result in eq. (5.22) explicitly illustrates the possible complications. Further we must add that even when eq. (6.5) is satisfied, the coefficient of the universal term may still vanish, depending on further details of the underlying theory. For example, if the bulk scalar theory, respects a discrete symmetry  $\Phi \rightarrow -\Phi$ , the coefficient will vanish unless  $m$  is an even integer. It is interesting to consider how these results would change if there were two or more relevant deformations with different conformal dimensions. We expect that in fact there would be no essential changes. The asymptotic expansions would have to be extended to allow separate factors  $\rho^{\alpha_i/2}$  from each of the deformations and the nonlinearities of the bulk theory would mix these terms. However, the schematic structure of the universal terms would remain as given in eq. (6.4) and the constraint on the conformal dimensions to produce a particular term would become

$$\sum m_i(d - \Delta_i) = d - 2 - 2n. \quad (6.6)$$

Of course, one contribution, which appears irrespective of the precise conformal dimension(s) of the relevant deformation(s), is the term with  $n = (d - 2)/2$ . As noted above, such contributions are known to appear for any even-dimensional CFT and the universal coefficients  $\gamma$  correspond to the central charges of the CFT [8, 3]. Even with the relevant deformation the present case, it is precisely these terms that appear with the central charges of the CFT that emerges in the UV regime. As was demonstrated in [6], the precise structure of these terms and their dependence on the geometry of the boundary metric and of the entangling surface can be derived using the PBH approach discussed in section 5. The latter was originally derived in the case

where the boundary theory was a pure CFT, however, all of the same contributions still appear in the asymptotic expansion when the relevant deformation is turned on. Hence the structure of this term does not change in the case where the boundary theory is deformed by a relevant deformation.

It is interesting that our holographic calculations indicate that for even  $d$ , the same central charges for the CFT emerging in the UV actually appear in the coefficients of all of the logarithmic contributions. This can be seen from the pre-factor of  $(L/\ell_P)^{d-1}$  which appears in all of our results. Since our bulk theory corresponds to Einstein gravity, all of the central charges are equal and we can not distinguish precisely which central charges appear in the various contributions. It may be interesting to repeat our analysis in the case where the bulk gravity theory includes higher curvature interactions since in principle, this would allow us to distinguish the different central charges [38] — see below.

The appearance of central charges in these new universal contributions hints at the close relation of these new terms with the trace anomaly. As is well known, with even  $d$ , EE in a CFT can be directly calculated using the trace anomaly, at least for geometries with sufficient symmetry [3, 8, 6, 9]. Typically, we consider the trace anomaly in a curved background, where it is usually related to various conformally invariant combinations of the curvature [24]. However, deforming the CFT with a relevant operator will also introduce additional terms in the trace anomaly related to the coupling to the new operators [15, 20]. While this situation has not been studied in detail, it is already evident that terms involving both the curvature and the coupling of the relevant operator appear in the trace anomaly in this situation. For example, the simplest such term, which arises for  $\Delta = (d+2)/2$ , takes the form [20]

$$\langle T^i_i \rangle = \frac{1}{2} \phi^{(0)} \left( \square + \frac{d-2}{4(d-1)} R \right) \phi^{(0)} + \dots \quad (6.7)$$

Given such a result, we can apply the approach, alluded to above, to calculate the entanglement entropy using the trace anomaly [3, 8, 6, 9] and we have confirmed that eq. (6.7) does indeed yield a universal contribution to the EE of precisely the form given in eq. (3.26). More generally, we expect that the new universal terms in eq. (6.4) are similarly related to new terms which the relevant operator induces in the trace anomaly. We hope to return to these issues elsewhere. Note that the calculations which we have sketched above apply to any general CFT with a relevant deformation and does not refer to holography. This would demonstrate that our results apply more broadly than to holographic field theories.

Another framework where these aspects of entanglement entropy are easily studied is with free field theories. In particular, ref. [1] considered a free massive scalar in

a flat background with a flat entangling surface. They found the logarithmic terms in eq. (1.2) appear for any even  $d \geq 4$  and these correspond to the  $n = 0$  terms in the general expression (6.4). It is amusing to compare this result to our holographic results which would correspond to a strongly coupled field theory. The natural relevant operator to consider would be one with  $\Delta = d - 2$ , as for a scalar mass term. In this case, the holographic contribution appears again for any even  $d$  but for  $d \geq 6$ . One can easily extend the free scalar calculation to examples where the geometry is curved [39] and logarithmic terms mixing curvatures and powers of the mass also appear in higher dimensions, similar to the results of our holographic study. Another simple extension of the free field calculations in [1] is to consider massive fermions [39]. In this case, it appears that various logarithmic contributions as in eq. (6.4) again arise in even dimensions. On the holographic side, it would be natural to compare to a relevant operator with  $\Delta = d - 1$ , as for a fermionic mass term. In this case, the holographic calculations yield a logarithmic contribution for any odd or even dimension with  $d \geq 4$ . Hence this discussion again indicates that the new universal terms (6.4), which we have uncovered here with holographic calculations, have a broader applicability and also arise in calculations of EE for more conventional field theories.

To close, let us observe that our analysis always assumed that the bulk gravitational theory was simply Einstein gravity and that the holographic EE was given by the standard Ryu-Takayanagi proposal (1.1). In various contexts, it would be interesting to consider the addition of higher curvature interactions to the bulk theory. The modification that such interactions would make in the holographic EE is not completely resolved. It is expected that eq. (1.1) would be replaced by the extremization of some geometric functional which produces the correct black hole entropy when evaluated on an event horizon. Recent progress was made in this direction for Lovelock theories of gravity [6, 29]. For example, with Gauss-Bonnet gravity in the bulk, eq. (1.1) is replaced by the following:

$$S(V) = \frac{2\pi}{\ell_P^{d-1}} \text{ext}_{v \sim V} \int_v d^{d-1}y \sqrt{h} [1 + \lambda L^2 \mathcal{R}] , \quad (6.8)$$

where  $\mathcal{R}$  denotes the Ricci scalar for the intrinsic geometry on  $v$  and  $\lambda$  is the (dimensionless) coupling which controls the strength of the curvature-squared interaction. While the appropriate entropy functional is not known for general higher curvature theories, we expect that it will have the form

$$S_{EE} = \frac{2\pi}{\ell_P^{d-1}} \text{ext}_{v \sim V} \int_v d^{d-1}y \sqrt{h} [1 + f(R, K^i, \Phi)] , \quad (6.9)$$

where  $f$  is some local scalar constructed from the bulk curvature  $R$ , the extrinsic curvature of the surface  $K^i$ , the bulk scalar  $\Phi$  (when a relevant deformation is introduced)

and derivatives of these building blocks. Note that eq. (6.8) can be re-expressed in this form using the Gauss-Codazzi equations [6]. Now the key observation is that in the FG gauge, any such scalar will admit an expansion in  $\tau$  beginning at order  $\tau^0$ . Hence the expansion of the full integrand begins with  $\tau^{-\frac{d}{2}}$ , just as in section 2. Further, the PBH transformations will continue to fix the asymptotic form of the asymptotic metric (and scalar), as well as the embedding functions, as discussed in section 5. The only change is that the constants appearing at various orders may take on new values as the equations of motion will have changed. In any event, as in the main text, any logarithmic contribution will only depend on the fixed boundary data and so we expect that the corresponding coefficient remains state independent when the bulk gravity theory is extended to include higher curvature interactions. Hence our previous result for the universal logarithmic contribution to the holographic EE is not changed in such a generalized holographic framework. Further we do not expect that the basic form of the universal terms will change in this scenario. However, it may be useful to examine these expressions in, *e.g.*, Lovelock gravity, as it may allow one to identify the specific central charge associated with the pre-factor  $(L/\ell_{\text{P}})^{d-1}$ , as discussed above.

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## A. Universality and Odd $d$

In many cases, no logarithmic contribution appears in the EE, *e.g.*, with a CFT in an odd number of spacetime dimensions. However, the constant term appearing in usual expansion in powers of the short-distance cut-off may still be regarded as a universal contribution to the EE [3]. The universality of this constant contribution is well-established for a variety of  $d = 3$  conformal quantum critical systems [25], as well as certain three-dimensional (gapped) topological phases [26]. However, from the discussion of the present paper, it is natural to expect that such a finite contribution will in fact depend on the details of the state in which the EE is calculated. Certainly in the holographic framework, this finite term should depend on  ${}^{(d/2)}g_{ij}$  and higher order

terms in the FG expansion. Hence in general, if we are to interpret this constant contribution to the EE as characteristic of the underlying field theory, we must also specify that the calculations were performed in the vacuum state of the theory.

In the following, we verify that the finite term in the EE does in fact depend on the state of the underlying theory with a simple holographic calculation. We consider a boundary CFT at finite temperature  $T$  and calculate the holographic EE across a spherical entangling surface of radius  $R$ . Working at low temperature, *i.e.*,  $RT \ll 1$ , we identify a temperature dependent contribution to the finite term in the EE. Note that this contribution is not simply the entropy density of the thermal bath multiplied by the volume of the ball bounded by the sphere — we comment on this point at the end of the appendix.

For simplicity, we consider the CFT in a flat  $d$ -dimensional spacetime and so at finite temperature, the holographic dual is a  $(d+1)$ -dimensional planar AdS black hole. The metric for this bulk solution can be written as

$$ds^2 = \frac{L^2}{z^2} \left( -f(z) dt^2 + dr^2 + r^2 d\Omega_{d-2}^2 \right) + \frac{L^2}{f(z)} \frac{dz^2}{z^2} , \quad (\text{A.1})$$

where we have introduced polar coordinates in the boundary directions and  $f(z)$  is given by

$$f(z) = 1 - \left( \frac{z}{z_+} \right)^d . \quad (\text{A.2})$$

Note that in this solution, the horizon appears at  $z = z_+$ . Further, in the limit  $z_+ \rightarrow \infty$ , we recover the AdS vacuum metric in Poincare coordinates. The Hawking temperature of this black hole solution is given by

$$T = \frac{d}{4\pi z_+} . \quad (\text{A.3})$$

To evaluate the holographic EE (1.1) for a spherical entangling surface, we must determine the extremal bulk surface described by a profile  $r = r(z)$  with the boundary condition  $r(z = 0) = R$ , as in section 3.2 except that the bulk space is now given by eq. (A.1). The induced metric on such a surface is given by

$$h_{\alpha\beta} dx^\alpha dx^\beta = \frac{L^2}{z^2} \left[ (r'^2 + 1/f) dz^2 + r^2 d\Omega_{d-2}^2 \right] , \quad (\text{A.4})$$

where the ‘prime’ denotes a derivative with respect to  $z$ . As a result the EE is given by

$$S = 2\pi \frac{L^{d-1}}{\ell_{\text{P}}^{d-1}} \Omega_{d-2} \int dz \frac{r^{d-2}}{z^{d-1}} \sqrt{r'^2 + 1/f} . \quad (\text{A.5})$$

In the case of pure AdS ( $z_+ = \infty$ ) the shape of the extremal surface can be obtained in the closed form given in eq. (3.22) [2, 3]. However, for general  $z_+$  we did not succeed to find a closed analytic expression and thus we proceed perturbatively in  $R/z_+ \ll 1$ . In light of eq. (A.3), this regime can be interpreted as a low temperature limit  $TR \ll 1$  (for a fixed number of dimensions  $d$ ). In this regime, the extremal surface will be close to that in eq. (3.22) and so we have  $z \lesssim R$ . Therefore we expand the integrand of eq. (A.5) in powers of  $\epsilon(z) = (z/z_+)^d$

$$S = 2\pi \frac{L^{d-1}}{\ell_P^{d-1}} \Omega_{d-2} \int_{\delta}^{z_{max}} dz \frac{r^{d-2}}{z^{d-1}} \sqrt{r'^2 + 1} \left( 1 + \frac{\epsilon(z)}{2(r'^2 + 1)} + \mathcal{O}(\epsilon^2) \right). \quad (\text{A.6})$$

Recall that with  $z_+ = \infty$ ,  $\epsilon(z) = 0$  and, according to eq. (3.22) (3.22), the profile of the extremal surface is given by  $r_0(z) = \sqrt{R^2 - z^2}$  and  $z_{max} = R$ . However, with  $\epsilon(z) \neq 0$ , both  $r(z)$  and  $z_{max}$  acquire corrections

$$r(z) = r_0(z) + \delta r(z), \quad z_{max} = R + \delta z_{max}, \quad (\text{A.7})$$

where these corrections are at least of order  $\epsilon$ . To solve for  $\delta r(z)$ , we may substitute eq. (A.7) into the action (A.6) and consider extremizing with respect to  $\delta r(z)$ . However, in doing so, we find that to leading order there are two contributions, one of order  $\delta r^2$  and the other of order  $\epsilon \delta r$ . Hence upon substituting the solution back into eq. (A.6), we would find that to leading order  $\delta r$  only makes contributions of  $\mathcal{O}(\epsilon^2)$  and so we can ignore this change in the profile. Similarly, the contribution to eq. (A.6) from the change  $\delta z_{max}$  involves evaluating the integrand (with  $r = r_0$ ) at  $z_{max} = R$  but this vanishes to leading order since  $r_0(R) = 0$ . Therefore if we work only to linear order in  $\epsilon(z)$ , we need only evaluate the second term in eq. (A.6) with the profile  $r_0(z)$ :

$$\delta S = 2\pi \frac{L^{d-1}}{\ell_P^{d-1}} \Omega_{d-2} \int_{\delta}^R dz \frac{r_0^{d-2}}{z^{d-1}} \frac{\epsilon(z)}{2\sqrt{r_0'^2 + 1}} + \mathcal{O}(\epsilon^2) = 2\pi \frac{L^{d-1}}{\ell_P^{d-1}} \frac{\Omega_{d-2}}{2(d+1)} \left( \frac{R}{z_+} \right)^d + \mathcal{O}(\delta^d, \epsilon^2). \quad (\text{A.8})$$

Hence combining this result with eq. (A.3), we find that  $\delta S \sim (RT)^d$  and so we have found a finite contribution to the EE which depends on the temperature (state) of the boundary CFT. Let us also note that for two-dimensional CFT's, one can get a closed expression for the EE at finite temperature [30] and expanding the latter in the limit of low temperature reproduces precisely our correction (A.8).

Let us consider extending the above calculation to a more general entangling surface to provide a general estimate for the contribution  $\delta S$  calculated above for a spherical surface. To make progress here, it is simplest to adopt the general framework and notation introduced in section 2. In particular, we begin by adopting the usual radial

coordinate of the FG expansion (2.1)

$$\begin{aligned} z &= L\rho^{1/2} \left( 1 + \frac{1}{4} \left( \frac{L}{z_+} \right)^d \rho^{d/2} \right)^{-2/d} \\ &\simeq L\rho^{1/2} \left( 1 - \frac{1}{2d} \left( \frac{L}{z_+} \right)^d \rho^{d/2} + \dots \right). \end{aligned} \quad (\text{A.9})$$

With this choice, the asymptotic expansion of the planar black hole metric (A.1) becomes

$$ds^2 \simeq \frac{L^2}{4\rho^2} d\rho^2 + \frac{1}{\rho} \left[ - \left( 1 - \frac{d-1}{d} \left( \frac{L}{z_+} \right)^d \rho^{d/2} \right) dt^2 + \left( 1 + \frac{1}{d} \left( \frac{L}{z_+} \right)^d \rho^{d/2} \right) \sum (dx^i)^2 \right]. \quad (\text{A.10})$$

Hence as expected, we see that the leading effect of the temperature appears in  $g_{ij}^{(d/2)}$  in the FG expansion (2.2). Now we choose some entangling surface  $\partial V$  in the flat boundary metric which is described by  $X^{(0)i}(y^a)$  and there will be some bulk surface described by the profile  $X^i(y^a, \tau)$  satisfying the boundary condition  $X^i(y^a, \tau = 0) = X^{(0)i}(y^a)$ . We make the same gauge choice as in eq. (2.6) and then the induced metric (2.5) becomes

$$\begin{aligned} h_{\tau\tau} &= \frac{L^2}{4\tau^2} \left( 1 + \frac{4\tau}{L^2} \partial_\tau X^i \partial_\tau X^j g_{ij} \right) \equiv \frac{L^2}{4\tau^2} \tilde{h}_{\tau\tau}, \\ h_{ab} &= \frac{1}{\tau} \partial_a X^i \partial_b X^j g_{ij} \equiv \frac{1}{\tau} \tilde{h}_{ab}. \end{aligned} \quad (\text{A.11})$$

As only the spatial coordinates are relevant here, we may write

$$g_{ij} \simeq \delta_{ij} (1 + \tilde{\epsilon}(\rho)) \quad \text{where} \quad \tilde{\epsilon}(\rho) = \frac{1}{d} \left( \frac{L}{z_+} \right)^d \rho^{d/2}. \quad (\text{A.12})$$

Now following the analysis above for the spherical entangling surface, we will expand the holographic EE in powers of  $\tilde{\epsilon}(\rho)$  and only keep the contribution that is linear in this term. As above, the deformation of the background metric will produce perturbations of the profile and the maximum value of  $\rho$ , both of which begin at linear order in  $\tilde{\epsilon}(\rho)$ :

$$X^i(y^a, \tau) = X_0^i(y^a, \tau) + \delta X^i(y^a, \tau), \quad \rho_{max} = \rho_{0,max} + \delta \rho_{max}. \quad (\text{A.13})$$

Again, to solve for  $\delta X^i(y^a, \tau)$ , we would extremize the area functional with respect to these functions. However, also as above, we find that to leading order there are two



contributions, one of order  $\delta X^{i^2}$  and the other of order  $\epsilon \delta X^i$ . Hence upon substituting the solution back into area, we would find that to leading order  $\delta X^i$  only makes contributions of  $O(\epsilon^2)$  and so we can ignore these changes in the profile. Similarly, the contribution from the change  $\delta \rho_{max}$  involves evaluating the integrand with  $X_0^i$  at  $\rho_{0,max}$  but this vanishes (to leading order) since by definition  $\rho_{0,max}$  is the point where the bulk surface (smoothly) closes off. Hence at this point, we have  $\sqrt{h_0}|_{\rho=\rho_{0,max}} = 0$ . Therefore if we work only to linear order in  $\epsilon(z)$ , we need only evaluate the variation to the area coming from the change of the background metric, given in eq. (A.12), with the profile  $X_0^i$ :

$$\begin{aligned} \delta S &= \frac{2\pi}{\ell_P^{d-1}} \oint_{\partial V} d^{d-2}y \int_{\delta}^{\rho_{max}} d\tau \frac{L}{2\tau^{d/2}} \sqrt{\tilde{h}_0} \left[ \frac{1}{2} \tilde{h}_0^{\alpha\beta} \delta_{\tilde{\epsilon}} \tilde{h}_{\alpha\beta} \right] \\ &= \frac{2\pi}{\ell_P^{d-1}} \oint_{\partial V} d^{d-2}y \int_{\delta}^{\rho_{max}} d\tau \frac{L}{4d} \sqrt{\tilde{h}_0} \left[ d - 1 - \frac{1}{(\tilde{h}_0)_{\tau\tau}} \right] \left( \frac{L}{z_+} \right)^d. \end{aligned} \quad (\text{A.14})$$

An essential feature of this result is that it is finite, *i.e.*, the leading factor of  $\tau^{-d/2}$  has been canceled by the  $\tau$ -dependence of  $\tilde{\epsilon}$ . In the vicinity of  $\tau = 0$ , the integrand reduces to essentially the  $\sqrt{\det^{(0)} h_{ab}}$ , *i.e.*, the area measure on  $\partial V$  in the boundary metric. Hence we would argue that the  $y$  integration essentially contributes a factor of  $\mathcal{A}_{d-2}$ , the area of the entangling surface. Certainly such a factor appears for entangling surfaces with sufficient symmetry, such as the spherical surface in the previous calculation. Similarly, the contribution of the  $\tau$  integral can be estimated to be roughly  $\rho_{max} \simeq \ell^2/L^2$ , where  $\ell$  is some characteristic scale of the geometry of  $\partial V$  which controls how far the extremal surface extends into the bulk. Again, for surfaces with sufficient symmetry, we can readily identify  $\ell$ . For example, in the above calculations,  $\ell$  is the radius  $R$  of the spherical entangling surface or in section 3.1,  $\ell$  would be the width of the slab with flat parallel boundaries. Therefore up to overall numerical factors, our estimate of this contribution to the holographic EE becomes

$$\delta S \simeq \frac{L^{d-1}}{\ell_P^{d-1}} \mathcal{A}_{d-2} \ell^2 T^d. \quad (\text{A.15})$$

Thus, our holographic calculations explicitly demonstrate that the constant contribution to the EE depends on the state in which the latter is calculated. Hence, while such a contribution certainly contains information that characterizes the underlying field theory, we must be careful in comparing various results to specify the state (*e.g.*, the vacuum) for which the calculations were performed.

Let us consider our holographic result (A.15) further. Given that the boundary CFT is at finite temperature  $T$ , the thermal bath will produce a uniform entropy density

$s \sim (L^{d-1}/\ell_{\text{P}}^{d-1}) T^{d-1}$  and so for a general region with volume  $\mathcal{V}_{d-1}$ , the corresponding thermal entropy would be  $\delta S_{\text{therm}} = (L^{d-1}/\ell_{\text{P}}^{d-1}) \mathcal{V}_{d-1} T^{d-1}$ . Hence comparing this result to eq. (A.15), we see that the finite temperature dependent contribution to the holographic EE which we have identified does not correspond to this thermal entropy. It may seem that we have found a discrepancy since we should expect that  $\delta S_{\text{therm}}$  should appear as a finite contribution in the EE [2, 3] and in fact, it seems that this contribution would dominate in the low temperature limit (given that  $\delta S_{\text{therm}}$  is proportional to a smaller power of  $T$ ). However, the latter limit provides the resolution of this apparent discrepancy. Since we are working in the limit  $\ell T \ll 1$ , the typical wavelength of the thermal excitations is much larger than the size of the entangling geometry and so it should not be a surprise that  $\delta S_{\text{therm}}$  has not appeared in our calculations. Instead this contribution would be expected to appear in the opposite limit  $\ell T \gg 1$ . In the latter case, the bulk surface would extend down to event horizon at  $z = z_+$  and  $\delta S_{\text{therm}}$  would naturally be produced by the portion of the extremal surface stretched along the horizon.

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